Convex Optimization and Applications

1 - Introduction

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Organization

- Contact:
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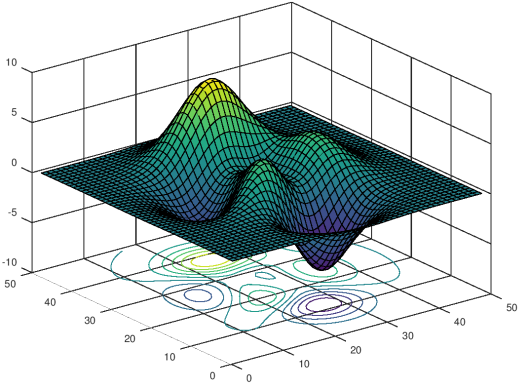
- Timing:
  - Monday 10-12, MA649
  - Thursday 14-16, MA550
  - Lecture every Monday
  - Lecture or Exercises every second week on Thursday
  - Occasionally: Programming exercises with python notebooks

Resources

- Handout put online as the lecture progresses
  `https://www.coga.tu-berlin.de/?206736`
- Slides for selected lectures
- The course is mainly based on the book
  "Convex Optimization",
  S. Boyd & L. Vandenberghe, 2004
  freely available online at `http://stanford.edu/~boyd/cvxbook/`.
- Selected chapters based on different resources
  (cf. website).

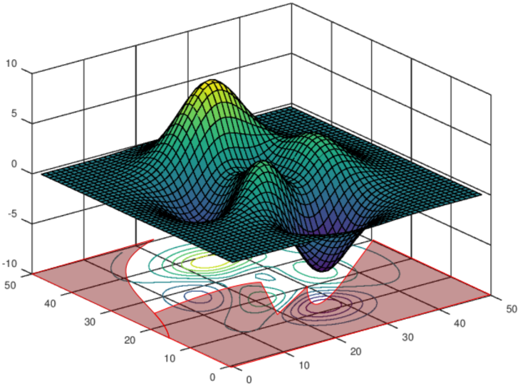
Optimization problem

$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad f_0(x)$$



Optimization problem

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\textbf{minimize}} \quad & f_0(\boldsymbol{x}) \\ s.t. \quad & f_i(\boldsymbol{x}) \leq 0, \forall i \in [m] \\ & h_j(\boldsymbol{x}) = 0, \forall j \in [p] \end{aligned}$$



Optimization problem

$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad f_0(\boldsymbol{x})$$
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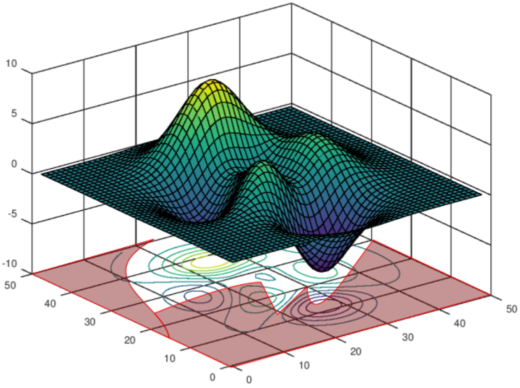


- Extremely versatile
- In general, a very hard problem

Optimization problem

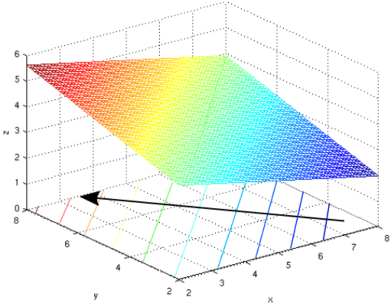
What makes an optimization problem difficult ?

- Number of variables
- Presence of constraints
- Non-convexity $\rightarrow$ Existence of local minima
- Discrete variables
  - This is, in fact, a special case of non-convex constraints:
  - $x_i \in \{0, 1\} \iff x_i^2 = x_i.$
  - $x_i \in \mathbb{Z} \iff \sin \dfrac{x_i}{\pi} = 0.$
- Regularity (smoothness) of functions.
  - existence of first or second-order derivatives, subgradients, ...

When everything is linear

Linear Programming

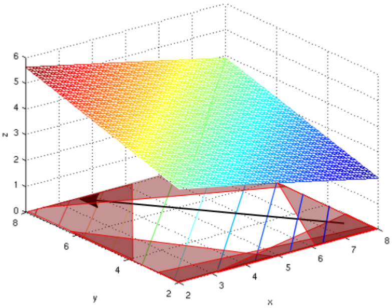
$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad \langle \boldsymbol{c}, \boldsymbol{x} \rangle$$



When everything is linear

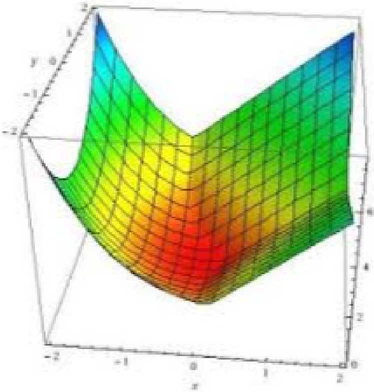
Linear Programming

$$
\begin{aligned}
\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\
s.t. \quad & \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq 0, \forall i \in [m] \\
& \langle \boldsymbol{a}_j, \boldsymbol{x} \rangle = 0, \forall j \in [p]
\end{aligned}
$$



Convex optimization

$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad f_0(x)$$



Convex optimization

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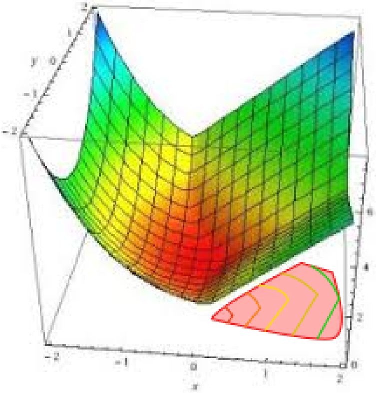
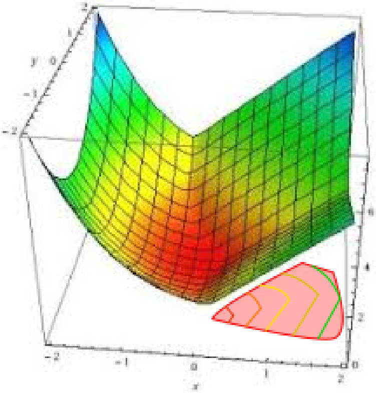


Convex optimization

As a nonlinear program:

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad & f_0(\boldsymbol{x}) \\ s.t. \quad & f_i(\boldsymbol{x}) \leq 0, \forall i \in [m] \\ & h_j(\boldsymbol{x}) = 0, \forall j \in [p] \end{aligned}$$



As a conic program:

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\ s.t. \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in K \end{aligned}$$

Why studying convex optimization ?

- Probably largest class of efficiently solvable problems
- Many applications ! Power of representation $>>$ LP
- “Locally, everthing is convex”
  - For many applications, finding a good local optimum is enough
  - This is often done by using convex optimization algos
- Convex relaxations of non-convex problems
  - Obtain lower bounds
  - Sometimes, rounding algorithms with a provable approximation guarantee

Objectives of this course

- Standard nonlinear programming approach vs. “modern” conic programming approach.
- Understand the concept of duality in convex optimization.
- Modelling issues: what can be (re)formulated as a convex optimization problem ?
- Review many applications of convex optimization
- Historical developments
- Learn to use modern python interfaces to solve conic problems with state-of-the art solvers in python
- Overview of Algorithms to solve convex problems
- Convex relaxations of non-convex (in particular, combinatorial) optimization problems

Evaluation

- Exercises
  - given 1 week in advance
  - check the exercises you've prepared. One student will be asked to explain his/her solution
  - You need 50% of all exercises *checked* to take the exam. (6-7 sessions, 2-3 ex./session $\rightarrow$ at least approx. 9 exercises)
- Oral examination.
  - Do not learn everything *by heart*.
  - However, you should know important definitions and have a rough idea of important proofs
  - You will not be asked to prove results from the courses.
  - Know which technical results exist and be able to find them in your handout if needed.
  - The focus is more on your general understanding, and your hability to solve new exercises