Exercise Sheet 6 (due date for Exercises 6.1 - 6.3: Feb. 06)

Exercise 6.1 (Homework)

A geometric interpretation of the Lasserre hierarchy for combinatorial optimization problems

Let $\boldsymbol{y} = (y_I)_{|I|\leq 2\delta}$ be a vector in $\mathcal{L}_{\delta-1}$ for some linear optimization problem with binary variables. In particular, it holds $y_\emptyset = 1$ and the moment matrix is positive definite: $M_\delta(\boldsymbol{y}) := (y_{I\cup J})_{|I|,|J|\leq\delta} \succeq 0$.

1. Show that there exists a collection of vectors $(\boldsymbol{v}_I)_{|I|\leq\delta}$ such that $\langle \boldsymbol{v}_I, \boldsymbol{v}_J\rangle = y_{I\cup J}, \qquad \forall |I|,|J| \leq \delta$.
2. Show that $\boldsymbol{v}_I$ belongs to the sphere of center $\frac{1}{2}\boldsymbol{v}_\emptyset$ and radius $\frac{1}{2}$.
3. More generally, if $J \subseteq I$, show that $\boldsymbol{v}_I$ belongs to the sphere of center $\frac{1}{2}\boldsymbol{v}_J$ and radius $\frac{\sqrt{y_J}}{2}$.
4. Let $I, J \subseteq [n]$ such that $y_I + y_J = 1$ and $y_{I\cup J} = 0$. What can you say about the geometric positions of $\boldsymbol{v}_I$ and $\boldsymbol{v}_J$?

Exercise 6.2 (Homework)

The S-lemma – Part 1

The goal of this exercise –and the following one– is to prove the *S-lemma*:

Let $A, B \in \mathbb{S}^n$ be symmetric matrices, and assume that $\exists \boldsymbol{x}_0 \in \mathbb{R}^n : \boldsymbol{x}_0^T A\boldsymbol{x}_0 > 0$. Then,

$$\forall \boldsymbol{x} \in \mathbb{R}^n : \Big(\boldsymbol{x}^T A\boldsymbol{x} \geq 0 \implies \boldsymbol{x}^T B\boldsymbol{x} \geq 0\Big) \iff \Big(\exists \lambda \geq 0 : B - \lambda A \succeq 0\Big). \tag{1}$$

1. One of the two directions of the equivalence (1) is easy to show. Which one ?
2. To prove the other implication, we consider the following SDP:

$$\begin{aligned} p^* = \inf \quad & \langle B, X\rangle \\ s.t. \quad & \langle A, X\rangle \geq 0 \\ & \langle I, X\rangle = 1 \\ & X \succeq 0. \end{aligned} \tag{SDP}$$

   Show that if Problem (SDP) is feasible, then it must have an optimal solution X^*.

3. Show that if $p^* = \langle B, X^*\rangle \geq 0$ and $\exists \boldsymbol{x}_0 : \boldsymbol{x}_0^T A\boldsymbol{x}_0 > 0$, then the right hand side of (1) holds.

Exercise 6.3 (Homework)

Proof of the S-lemma – Part 2

1. Let X^* be the optimal solution of Problem (SDP) (cf. Exercise 6.2). Define $\bar{A} = (X^*)^{1/2}A(X^*)^{1/2}$ and $\bar{B} = (X^*)^{1/2}B(X^*)^{1/2}$, and consider an eigenvalue decomposition $\bar{A} = U\Lambda U^T$. Show that if $\boldsymbol{z} \in \{-1,1\}^n$, then $(U\boldsymbol{z})^T\bar{A}(U\boldsymbol{z}) \geq 0$.
2. Compute the value of the matrix $Z = \frac{1}{2^n}\sum_{\boldsymbol{z}\in V} \boldsymbol{z}\boldsymbol{z}^T$, where the sum goes over the 2^n vertices.
   ($V = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_{2^n}\}$) of the hypercube $\{-1,1\}^n$. *Hint: You can use a statistical reasoning.*

3. Compute the value of the quantity
$$\frac{1}{2^n}\sum_{\boldsymbol{z}\in V}(U\boldsymbol{z})^T\bar{B}(U\boldsymbol{z}),$$
and conclude that if the condition at the LHS of (1) holds, then we have $p^*\geq 0$.

Exercise 6.4

Consider the pair of conic programs

$$(P):\ \textbf{minimize}\{\boldsymbol{c}^T\boldsymbol{x}:\quad A\boldsymbol{x}\preceq_K \boldsymbol{b}\}\qquad\qquad (D):\ \textbf{maximize}\{-\boldsymbol{b}^T\boldsymbol{z}:\quad A^T\boldsymbol{z}+\boldsymbol{c}=\boldsymbol{0},\ \boldsymbol{z}\succeq_{K^*}\boldsymbol{0}\}$$

The self-dual embedding is an elegant technique which allows one to *glue* together (P) and (D), in a form which is strictly feasible, and such that optimal solutions can be transformed to either an optimal solution of (P) and (D), or to obtain a certificate of primal or dual infeasibility. In this exercise, we study a simplified form of this embedding:

$$\begin{aligned}\underset{\boldsymbol{s},\kappa,\boldsymbol{x},\boldsymbol{z},\tau}{\textbf{minimize}}\quad & 0 \\ s.t.\quad & \begin{bmatrix}0\\ \boldsymbol{s}\\ \kappa\end{bmatrix}=\begin{pmatrix}0 & A^T & \boldsymbol{c}\\ -A & 0 & \boldsymbol{b}\\ -\boldsymbol{c}^T & -\boldsymbol{b}^T & 0\end{pmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{z}\\ \tau\end{bmatrix}\\ & \tau\geq 0,\ \kappa\geq 0,\ \boldsymbol{s}\succeq_K\boldsymbol{0},\ \boldsymbol{z}\succeq_{K^*}\boldsymbol{0}\end{aligned}\qquad\text{(PD)}$$

1. Show that Problem (PD) is self-dual, i.e., the dual of (PD) is equivalent to (PD).
2. Show that any feasible solution of (PD) satisfies $\boldsymbol{z}^T\boldsymbol{s}+\tau\kappa=0$, and explain why (PD) is feasible, but not strictly feasible (or even essentially strictly feasible).

(In practice, note that it is possible to add auxiliary variables and constraints in the problem in order to make it strictly feasible, and such that the following results still hold).

3. Assume that $(\boldsymbol{s},\kappa,\boldsymbol{x},\boldsymbol{z},\tau)$ is feasible for (PD), with $\tau>0$. Show how to construct a pair of primal and dual optimal solutions for (P) and (D).
4. Assume that $(\boldsymbol{s},\kappa,\boldsymbol{x},\boldsymbol{z},\tau)$ is feasible for (PD), with $\kappa>0$. Show that either $\boldsymbol{c}^T\boldsymbol{x}<0$ or $\boldsymbol{b}^T\boldsymbol{z}<0$ must hold, and how to construct a certificate of dual infeasibility in the former case, or a certificate of primal infeasibility in the latter case.

Exercise 6.5

Consider a symmetric polynomial matrix $M\in(\mathbb{R}_{2d}[x_1,\ldots,x_n])^{m\times m}$ of degree $2d$ on n variables (i.e., $\boldsymbol{x}\mapsto M_{ij}(\boldsymbol{x})=M_{ji}(\boldsymbol{x})\in\mathbb{R}_{2d}[x_1,\ldots,x_n],\forall i,j\in[m]$). We say that M is an *SOS-matrix* iff $\exists$ a polynomial matrix $H\in(\mathbb{R}_d[x_1,\ldots,x_n])^{m\times m}$ such that $M(\boldsymbol{x})=H(\boldsymbol{x})H(\boldsymbol{x})^T$.

1. Show that $M(\boldsymbol{x})$ is an SOS-matrix iff the polynomial
$$(\boldsymbol{x},\boldsymbol{y})\mapsto\boldsymbol{y}^TM(\boldsymbol{x})\boldsymbol{y}\in\mathbb{R}_{2d+2}[x_1,\ldots,x_n,y_1,\ldots,y_n]$$
is a sum of squares.
2. A polynomial p is called *SOS-convex* if its Hessian matrix $\nabla^2p(\boldsymbol{x})$ is an SOS-matrix. Show that an SOS-convex polynomial is convex.