Exercise Sheet 5 (due date for Exercises 5.1 - 5.3: Jan. 16)

Exercise 5.1 (Homework)

The partition problem is defined as follows: Given integers $a_1, \ldots, a_n$, does there exist a subset $S \subseteq [n]$ such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$?

An optimization version of this problem is the following:

$$\underset{S \subseteq [n]}{\textbf{minimize}} \left(\sum_{i \in S} a_i - \sum_{i \notin S} a_i \right)^2$$

1. Reformulate the problem as a binary quadratic program and formulate an SDP relaxation.
2. Show how to change the relaxation to handle the constraint $|S| = k$.

Exercise 5.2 (Homework)

Copositive programming formulation of the maximum independent set

A conic optimization problem involving the cone $\mathcal{C}_n$ (or its dual) is called a *copositive program*. The goal of this exercise is to show that the maximum independent set of a graph can be computed by solving a copositive program. Hence copositive programming is intractable, but it is known that copositive programs can be *approximated* by SDPs. Let $G = (V, E)$ be a simple graph with n vertices. We recall that the stability number satisfies $\alpha(G) \leq \vartheta(G)$, where $\vartheta(G)$ is the Lovasz-theta number of G:

$$\begin{aligned} \vartheta(G) = \max_X \quad & \langle J, X \rangle \\ s.t. \quad & \langle I, X \rangle = 1 \\ & X_{ij} = 0, \qquad \forall ij \in E \\ & X \succeq_{\mathbb{S}_+^n} 0. \end{aligned}$$

In what follows, we will show that $\alpha(G) = \vartheta^*(G)$, where $\vartheta^*(G)$ is the value of the copositive program obtained by replacing the constraint "$X \succeq_{\mathbb{S}_+^n} 0$" by "$X \succeq_{\mathcal{C}_n^*} 0$".

Recall the following definitions: $\boldsymbol{x} \in K$ is an *extreme ray* of a convex cone K if the only possibility to express $\boldsymbol{x}$ as a barycenter of two other rays $\boldsymbol{y}, \boldsymbol{z} \in K$ is to take $\boldsymbol{y} = \alpha \boldsymbol{x}$ and $\boldsymbol{z} = \beta \boldsymbol{x}$ for some scalars α and β. Similarly, $\boldsymbol{x} \in S$ is an *extreme point* of a convex set S if the only possibility to express $\boldsymbol{x}$ as a barycenter of two other points $\boldsymbol{y}, \boldsymbol{z} \in S$ is to take $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}$.

You can use (without proof) the following result:

Let K be a convex cone, and let $H = \{\boldsymbol{x} : \boldsymbol{a}^T \boldsymbol{x} = b\}$ be an hyperplane, with $\boldsymbol{a} \in \textbf{int} K^*$. Then it holds

$$\textbf{ext-points}(K \cap H) = \textbf{ext-rays}(K) \cap H.$$

1. Define $\mathcal{K} = \{X \in \mathcal{C}_n^* : \quad \forall ij \in E, X_{ij} = 0\}$, and observe that $\mathcal{K}$ is a cone. Show that X is an extreme ray of $\mathcal{K}$ iff $X = \boldsymbol{x}\boldsymbol{x}^T$ for some $\boldsymbol{x} \in \mathbb{R}_+^n$ supported by a stable set of G (i.e., $S = \{i : x_i \neq 0\}$ is stable).
2. Use the result of 1. to identify the set of extreme points of the feasible set of the copositive program for $\vartheta^*(G)$. Conclude that $\vartheta^*(G) = \alpha(G)$.

Exercise 5.3 (Homework)

We will give another proof of the copositive programming formulation for $\alpha(G)$ (i.e., $\vartheta^*(G) = \alpha(G)$) relying on the Motzkin-Straus theorem, which can be stated as follows (Motzkin and Straus originally stated an equivalent result involving the clique number $\omega(G)$):

Let $G = (V, E)$ be a simple grah with n vertices. Then, $1/\alpha(G)$ is the minimum of a (nonconvex) quadratic over the probability simplex $\Delta_n^=$:

$$\frac{1}{\alpha(G)} = \min_{\boldsymbol{x}} \Big\{ \boldsymbol{x}^T (I + A)\boldsymbol{x} : \ \boldsymbol{x} \geq \mathbf{0},\ \mathbf{1}^T \boldsymbol{x} = 1 \Big\},$$

where A is the adjacency matrix of G, i.e. the $\{0,1\}$-symmetric matrix such that $A_{ij} = 1 \iff ij \in E$. :

1. We recall the definition

$$\vartheta^*(G) := \max_X \quad \Big\{ \langle J, X\rangle : \langle I, X\rangle = 1,\ X \succeq_{\mathcal{C}_n^*} 0,\ X_{ij} = 0,\ \forall ij \in E \Big\}.$$

   Show that $\vartheta^*(G) \geq \alpha(G)$.

To show the converse inequality, we consider the dual program defining $\vartheta^*(G)$. By dualizing the copositive program from the previous exercice as in the lecture, but dualizing over $\mathcal{C}_n^*$ instead of $\mathbb{S}_+^n$, we can show (*you don't have to prove this*):

$$\vartheta^*(G) = \min_{t,Z} \Big\{ t : \ tI + Z \succeq_{\mathcal{C}^n} J, \quad Z_{ij} = 0, \forall ij \in \bar{E},\ Z_{ii} = 0, \forall i \in [n] \Big\}.$$

2. Show that there is a solution to the above program in which all nonzero elements of Z are equal, and hence,

$$\vartheta^*(G) = \min_{t,z \in \mathbb{R}} \Big\{ t : \ tI + zA \succeq_{\mathcal{C}^n} J \Big\}.$$

3. By the Motzkin-Straus theorem (cf. text at the beginning of this exercise), we have

$$\frac{1}{\alpha(G)} \leq \boldsymbol{x}^T (I + A)\boldsymbol{x}, \quad \forall \boldsymbol{x} \in \Delta_n^=.$$

   Show that this implies $\frac{1}{\alpha(G)} J \preceq_{\mathcal{C}_n} I + A$, and then show that $\vartheta^*(G) \leq \alpha(G)$.

Exercise 5.4

In the lecture, we studied the SDP relaxation of MAXCUT

$$\begin{aligned} p^* = \max_X \quad & \frac{1}{4}\langle L, X\rangle \\ s.t. \quad & \operatorname{diag}(X) = \mathbf{1} \\ & X \succeq 0 \end{aligned}$$

where $L := \sum_{ij \in E} w_{ij} (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^T$ is the Laplacian matrix of G.

1. Show that the dual of this SDP is

$$\begin{aligned} d^* = \min_{\boldsymbol{y}} \quad & \frac{1}{4}\ \mathbf{1}^T \boldsymbol{y} \\ s.t. \quad & L \preceq \operatorname{Diag}(\boldsymbol{y}) \end{aligned}$$

2. Does strong duality hold?

3. Show that

$$\operatorname{maxcut}(G) \leq \frac{n}{4} \lambda_{\max}(L).$$

Exercise 5.5

We return to Motzkin-Straus theorem studied in Exercise 5.3. Let $Q := I + A$. After a simple change of variable (substitute $\boldsymbol{x} \geq \boldsymbol{0}$ by $\boldsymbol{x} \circ \boldsymbol{x} = (x_i^2)_{i \in [n]}$), we can formulate $1/\alpha(G)$ as the minimum value of a quartic (a polynomial of degree 4) over the unit sphere:

$$\frac{1}{\alpha(G)} = \min \Big\{ \sum_{i,j} Q_{i,j}\, x_i^2 x_j^2 : \quad \sum_i x_i^2 = 1 \Big\}.$$

1. Write down the first level ($\delta = 0$) of the Lasserre hierarchy for the above problem: Explain what is the dimension of the vector of moments $\boldsymbol{y}$, and give the coordinates of the moment and localizing matrices.

2. Simplify the LMI constraints involving the localizing matrix as much as you can. You can use the notation

$$Y_i = y_{\boldsymbol{e}_i}, \quad Y_{ij} = y_{\boldsymbol{e}_i + \boldsymbol{e}_j}, \quad Y_{ijk} = y_{\boldsymbol{e}_i + \boldsymbol{e}_j + \boldsymbol{e}_k}, \quad Y_{ijk\ell} = y_{\boldsymbol{e}_i + \boldsymbol{e}_j + \boldsymbol{e}_k + \boldsymbol{e}_\ell}.$$