Exercise Sheet 4 (due date for homework exercises 4.1 to 4.3: Dec. 5)

Exercise 4.1 (Homework)

1. Show that the Lagrangian dual of the following SDP

$$\begin{aligned} p^* = \max \quad & \langle A, X\rangle \\ s.t. \quad & \mathbf{tr}\ X = 1 \\ & X \succeq 0. \end{aligned}$$

   is equivalent to

$$\begin{aligned} d^* = \min_{\lambda\in\mathbb{R}} \quad & \lambda \\ s.t. \quad & \lambda I \succeq A. \end{aligned}$$

2. What does the weak duality theorem tell us about p^* and d^*?

3. Give a closed-form expression for the value of d^*.

4. How can the strong duality theorem be applied?

5. Find a feasible X such that $\langle A, X\rangle = d^*$. *Hint:* You can search for a matrix X of the form $X = \boldsymbol{x}\boldsymbol{x}^T$ for some well chosen vector $\boldsymbol{x} \in \mathbb{R}^n$.

Exercise 4.2 (Homework)

We consider the non-convex least-squares approximation problem with binary constraints

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 \\ s.t. \quad & x_k \in \{-1, 1\}, \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. We assume that $\mathbf{rank} A = n$, i.e., $A^T A$ is nonsingular.

1. Derive the Lagrangian dual of this problem, when each constraint $x_k \in \{-1, 1\}$ is replaced by $x_k^2 = 1$. You can ignore the case in which $A^T A + diag(\boldsymbol{\mu})$ is positive semidefinite but singular.

2. Reformulate the Lagrangian dual as an SDP. Can we assert that strong duality holds for this problem? (You can assume that the SDP you derived is still valid in the singular case.)

Exercise 4.3 (Homework)

Derive the Lagrangian dual of the following SOCP:

$$\begin{aligned} \min_{x,y} \quad & x + y \\ s.t. \quad & \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| \leq 1 \\ & \left\| \begin{bmatrix} x-1 \\ y \end{bmatrix} \right\| \leq 1. \end{aligned}$$

Exercise 4.4

Consider the problem

$$\begin{array}{ll} \min & x_2 \\ s.t. & \begin{pmatrix} x_2+1 & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & x_2 & 0 \end{pmatrix} \succeq 0. \end{array}$$

1. Derive the Lagrangian dual of this problem.
2. Compute p^* and d^*.

Exercise 4.5

We consider the SDP

$$\begin{array}{ll} \underset{\boldsymbol{x}\in\mathbb{R}^3}{\textbf{minimize}} & x_1 - 2x_3 \\ & x_1 + x_2 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \succeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Show how to reformulate this SDP under the *standard form*: $\min\langle C, X\rangle,\ s.t.\ \langle A_i, X\rangle = b_i,\ X \succeq 0$. (You do not have to give the complete SDP, but at least an example for each type of constraint of the resulting SDP).

Exercise 4.6

Consider the optimization problem

$$\begin{array}{rl} p^* = \min & x^2 + 1 \\ s.t. & (x-2)(x-4) \leq 0 \end{array}$$

with $x \in \mathbb{R}$.

1. Give the feasible set, optimal value, and optimal solution.
2. Derive the Lagrange dual function and state the dual problem. Find the dual optimal value d^* and the dual optimal solution λ^*. Does strong duality hold?
3. *Sensitivity analysis.* Let $p^*(u)$ denote the optimal value of the problem

   $$\begin{array}{rl} p^*(u) = \min & x^2 + 1 \\ s.t. & (x-2)(x-4) \leq u, \end{array}$$

   as a function of the parameter u. Give $p^*(u)$ in a closed-form expression and verify that

   $$\left.\frac{dp^*(u)}{du}\right|_{u=0} = -\lambda^*$$