Exercise Sheet 3 (due date for homework exercises 3.1 to 3.3: Nov. 21)

Exercise 3.1 (Homework)

For a non-square matrix $Y \in \mathbb{R}^{m\times k}$, the norm of Y induced by the Euclidean norm is the spectral norm, defined by $\|Y\| = \sqrt{\lambda_{\max}(YY^T)}$.

1. Show that $\|Y\| \leq t \iff \begin{bmatrix} t\,I_m & Y \\ Y^T & t\,I_k \end{bmatrix} \succeq 0$

Now, let P be the polyhedron $\{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\}$ and define $F(\boldsymbol{x}) = F_0 + x_1F_1 + x_2F_2 + \ldots + x_nF_n$, and $G(\boldsymbol{x}) = G_0 + x_1G_1 + x_2G_2 + \ldots + x_nG_n$, where for all i, $F_i \in \mathbb{R}^{m\times k}$ and $G_i \in \mathbb{S}^m$. Show how to pose the following problems as SDPs (You don't need to put the SDP in a standard form):

2. **minimize** $\|F(\boldsymbol{x})\|$ over $\boldsymbol{x} \in P$.
3. **minimize** $\lambda_{\max}(G(\boldsymbol{x})) - \lambda_{\min}(G(\boldsymbol{x}))$ over $\boldsymbol{x} \in P$.
4. **minimize** $\sum_{i=1}^m |\lambda_i(\boldsymbol{x})|$ over $\boldsymbol{x} \in P$, where $\lambda_1(\boldsymbol{x}) \geq \cdots \geq \lambda_m(\boldsymbol{x})$ are the eigenvalues (counted with multiplicity) of $G(\boldsymbol{x})$.
   Hint: You can use the following result: If G is a symmetric matrix, it admits a decomposition of the form $G = G_+ - G_-$, where $G_+ \succeq 0$ and $G_- \succeq 0$, and which satisfies the following properties:
   (i) The nonzero eigenvalues of G^+ are the positive eigenvalues of G;
   (ii) The nonzero eigenvalues of G^- are the absolute values of the negative eigenvalues of G;
   (iii) If $G = C - D$ for another decomposition with $C \succeq 0, D \succeq 0$, then we have $\operatorname{trace} C \geq \operatorname{trace} G_+$, and $\operatorname{trace} D \geq \operatorname{trace} G_-$.

Exercise 3.2 (Homework)

Formulate the problem of maximizing $f(\boldsymbol{x})$ over the polyhedron $P = \{\boldsymbol{x} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ (we assume that P is nonempty) as an SOCP, where

1. $f(\boldsymbol{x}) = n(\sum_i \frac{1}{x_i})^{-1}$ is the harmonic mean of $\boldsymbol{x}$. To do so, show the following:
   (a) Let $t \geq 0$. We have $f(\boldsymbol{x}) \geq t$ if and only if there exists $\boldsymbol{\mu} \geq 0$ s.t.
   $$\begin{cases} t^2 \leq \mu_i x_i n & \forall i = 1, \ldots, n \\ t \geq \sum_{i=1}^n \mu_i. \end{cases}$$
   (b) Conclude.
2. $f(\boldsymbol{x}) = (\prod_i x_i)^{\frac{1}{n}}$ is the geometric mean of $\boldsymbol{x}$. To do so, show the following:
   (a) Let $t \geq 0$. We have $(x_1x_2x_3x_4)^{\frac{1}{4}} \geq t$ if and only if $\exists u, v \geq 0$ s.t.
   $$\begin{cases} u^2 \leq x_1x_2 \\ v^2 \leq x_3x_4 \\ t^2 \leq uv. \end{cases}$$
   (b) Explain briefly how (a) can be extended to any n being a power of two.
   (c) How can this be extended to any positive integer n?

Exercise 3.3 (Homework)

Show that the following functions have a K_{exp}-representation, where K_{exp} denotes the exponential cone:

1. The relative entropy (Kullback-Leibler divergence) of two non-negative vectors $\boldsymbol{x}, \boldsymbol{y} \geq 0$:

$$f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} x_i \log \frac{x_i}{y_i},$$

   with the convention discussed in the lecture, when the variables attain the value 0.

2. The log-sum-exp function: $g(\boldsymbol{x}) = \log(\sum_{i=1}^{n} e^{x_i})$

Exercise 3.4

We consider the SDP

$$\begin{aligned}
\underset{X \in \mathbb{S}^3}{\textbf{minimize}} \quad & \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 6 \end{pmatrix}, X \right\rangle \\
& \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, X \right\rangle = 2 \\
& \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, X \right\rangle = -1 \\
& X \succeq 0.
\end{aligned}$$

Reformulate this SDP under an equivalent *conic programming form*: $\min_{\boldsymbol{x}} \boldsymbol{c}^T \boldsymbol{x},\ s.t.\ F\boldsymbol{x} = \boldsymbol{g},\ \sum_i x_i A_i \succeq B$.