Exercise Sheet 2 (due date for homework exercises 2.1 to 2.3: Nov. 7)

Exercise 2.1 (Homework)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, with $\mathbf{dom} f = \mathbb{R}^n$. Moreover, let f be bounded from above on $\mathbb{R}^n$. Show that f is constant.

Exercise 2.2 (Homework)

We consider the optimization problem

$$\begin{aligned} \textbf{minimize} \quad & f_0(\boldsymbol{x}) \\ s.t. \quad & G\boldsymbol{x} \leq \boldsymbol{h} \\ & A\boldsymbol{x} = \boldsymbol{b}, \end{aligned} \tag{LFP}$$

where

$$f_0(\boldsymbol{x}) = \frac{\boldsymbol{c}^T\boldsymbol{x} + d}{\boldsymbol{e}^T\boldsymbol{x} + f}, \qquad \mathbf{dom} f_0 = \{\boldsymbol{x} |\ \boldsymbol{e}^T\boldsymbol{x} + f > 0\}.$$

We assume that the feasible set $\{\boldsymbol{x} |\ G\boldsymbol{x} \leq \boldsymbol{h}, A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{e}^T\boldsymbol{x} + f > 0\}$ is nonempty. The goal of the exercise is to show that using the change of variables

$$\boldsymbol{y} = \frac{\boldsymbol{x}}{\boldsymbol{e}^T\boldsymbol{x} + f}, \qquad z = \frac{1}{\boldsymbol{e}^T\boldsymbol{x} + f}$$

the Problem (LFP) is "equivalent" to the following problem (LP):

$$\begin{aligned} \min \quad & g_0(\boldsymbol{y}, z) = \boldsymbol{c}^T\boldsymbol{y} + dz \\ s.t. \quad & G\boldsymbol{y} - \boldsymbol{h}z \leq \boldsymbol{0} \\ & A\boldsymbol{y} - \boldsymbol{b}z = \boldsymbol{0}, \\ & \boldsymbol{e}^T\boldsymbol{y} + fz = 1 \\ & z \geq 0, \end{aligned} \tag{LP}$$

with variables $\boldsymbol{y} \in \mathbb{R}^n, z \in \mathbb{R}$. More precisely, show that:

1. If $\boldsymbol{x}$ is feasible for (LFP), then $(\boldsymbol{y}, z)$ is feasible for (LP), and $f_0(\boldsymbol{x}) = g_0(\boldsymbol{y}, z)$.
2. If $\boldsymbol{y}, z$ is feasible for (LP) and $z \neq 0$, there exists feasible $\boldsymbol{x}$ for (LFP) such that $f_0(\boldsymbol{x}) = g_0(\boldsymbol{y}, z)$.
3. If $\boldsymbol{y}, z$ is feasible for (LP) and $z = 0$, then you can find a sequence of feasible $\boldsymbol{x}_i$'s for (LFP) such that $\lim_{i\to\infty} f_0(\boldsymbol{x}_i) = g_0(\boldsymbol{y}, z)$.

Exercise 2.3 (Homework)

Show that the separation problem can be solved for the non-negative orthant $\mathbb{R}^n_+$, the Lorentz cone $\mathbb{L}^n_+$ and the positive semidefinite cone $\mathbb{S}^n_+$, i.e., either assert that an element x is in the cone K or explicitly give a hyperplane separating x and K, that is, find $h \neq 0$ such that

$$\langle h, y \rangle \geq \langle h, x \rangle \quad \forall y \in K.$$

Reminder: $\mathbb{L}^n_+ := \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|x\|_2 \leq t\}$

Exercise 2.4

Consider the optimization problem

$$\begin{aligned} \textbf{minimize} \quad & f_0(x_1, x_2) \\ s.t. \quad & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & \boldsymbol{x} \geq \mathbf{0}. \end{aligned}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value:

1. $f_0(x_1, x_2) = x_1 + x_2$
2. $f_0(x_1, x_2) = -x_1 - x_2$
3. $f_0(x_1, x_2) = x_1$
4. $f_0(x_1, x_2) = \max(x_1, x_2)$
5. $f_0(x_1, x_2) = x_1^2 + 9x_2^2$