Exercise Sheet 1 (due date for homework exercises 1.1 to 1.3: Oct. 24)

Exercise 1.1 (Homework)

Let $A \in \mathbb{S}^n_{++}$ and $B \in \mathbb{S}^n$.

1. Show that for all matrices U of appropriate size, $A \preceq B \implies U^T A U \preceq U^T B U$.
2. Show that $A \preceq \boldsymbol{I}_n \iff A^{-1} \succeq \boldsymbol{I}_n$. (Hint: Use the matrix $A^{\frac{1}{2}}$)
3. Show that $A \preceq B$ implies that B is invertible, and $A^{-1} \succeq B^{-1}$.

   We define the ellipsoid $\mathcal{E}_A := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T A^{-1} \boldsymbol{x} \leq 1\}$.
4. Let now B be positive definite. Conclude that $A \preceq B \iff \mathcal{E}_A \subseteq \mathcal{E}_B$.

Exercise 1.2 (Homework)

1. Let $A_1 \in \mathbb{R}^{m \times n}$ and $A_2 \in \mathbb{R}^{n \times n}$ invertible. We define

   $$S_1 := \{A_1 x : x \geq 0\} \quad \text{and} \quad S_2 := \{x : A_2 x \geq 0\}.$$

   Determine the dual cones of S_1 and S_2 and give it in the form

   $$\{y : By \geq 0\}.$$

   for some suitable B.
2. Consider $K := \{x : x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}$. Determine the dual cone of K.

Exercise 1.3 (Homework)

Let K, K_1 and K_2 be cones. Show the following properties:

1. K^* is a closed convex cone.
2. $K_1 \subseteq K_2 \implies K_2^* \subseteq K_1^*$.
3. $(K_1 \times K_2)^* = K_1^* \times K_2^*$

   Now, let K be additionally closed and convex.
4. $K^{**} = K$
5. K has a nonempty interior $\iff$ K^* pointed.

Exercise 1.4

Let $C \subseteq \mathbb{R}^n$ with nonempty interior. Moreover, let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, and $g : \mathbb{R}^n \to \mathbb{R}$ be concave, with $\mathbf{dom} f = \mathbf{dom}\, g = C$. Suppose that the inequality $g(\boldsymbol{x}) \leq f(\boldsymbol{x})$ holds everywhere on C. Show that we can fit an affine function between f and g, i.e., there exists an affine function h such that

$$\forall \boldsymbol{x} \in C, \quad g(\boldsymbol{x}) \leq h(\boldsymbol{x}) \leq f(\boldsymbol{x}).$$

Exercise 1.5

Let K be a proper cone. Show that the generalized inequality $\preceq_K$ satisfies the following properties:

1. transitivity: $\boldsymbol{x} \preceq_K \boldsymbol{y}$ and $\boldsymbol{y} \preceq_K \boldsymbol{z} \implies \boldsymbol{x} \preceq_K \boldsymbol{z}$
2. reflexivity: $\boldsymbol{x} \preceq_K \boldsymbol{x}$.
3. antisymmetry: $\boldsymbol{x} \preceq_K \boldsymbol{y}$ and $\boldsymbol{y} \preceq_K \boldsymbol{x} \implies \boldsymbol{x} = \boldsymbol{y}$.
4. preservation under addition: $\boldsymbol{x} \preceq_K \boldsymbol{y}$ and $\boldsymbol{u} \preceq_K \boldsymbol{v} \implies \boldsymbol{x} + \boldsymbol{u} \preceq_K \boldsymbol{y} + \boldsymbol{v}$.
5. preservation under nonnegative scaling: $\boldsymbol{x} \preceq_K \boldsymbol{y}$ and $\alpha \geq 0 \implies \alpha\boldsymbol{x} \preceq_K \alpha\boldsymbol{y}$.