Convex Optimization and Applications

12 - Advanced topics in convex optimization

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Outline

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From Copositive Programming to Nonnegative Polynomials

$$X \succeq_{\mathcal{C}_n} 0 \iff \boldsymbol{u}^T X \boldsymbol{u} \geq 0, \quad \forall \boldsymbol{u} \geq \boldsymbol{0}$$

$$\iff \text{The polynomial } \boldsymbol{u} \mapsto \sum_{ij} X_{ij} u_i u_j \text{ is nonnegative over } \mathbb{R}^n_+.$$

$$\iff \text{The polynomial } \boldsymbol{u} \mapsto \sum_{ij} X_{ij} u_i^2 u_j^2 \text{ is nonnegative over } \mathbb{R}^n.$$

And we can test the nonnegativity of polynomials using hierachies of SDP !

Univariate polynomials

We identify polynomials of degree d with their vector of coefficients:

$$p(x) = \sum_{i=0}^{d} p_i x^i = \boldsymbol{p}^T \bar{\boldsymbol{x}}, \qquad \text{with } \bar{\boldsymbol{x}} := [1, x, x^2, \ldots, x^d]^T.$$

Definition

We define the cone of nonnegative polynomials of degree $\leq n$

$$P_{2d}^{\text{pos}} := \{\boldsymbol{p} \in \mathbb{R}^{2d+1} : \sum_{i=0}^{2d} p_i\, x^i \geq 0, \quad \forall x \in \mathbb{R}\} \subset \mathbb{R}^{2d+1}$$

and the cone of *sum of squares*

$$P_{2d}^{\text{sos}} := \{\boldsymbol{p} \in \mathbb{R}^{2d+1} : \exists \boldsymbol{q}_1, \ldots, \boldsymbol{q}_r \in \mathbb{R}^{d+1}, \sum_{i=0}^{2d} p_i\, x^i = \sum_{j=1}^{r} (\boldsymbol{q}_j^T \bar{\boldsymbol{x}})^2\}$$

These two cones are proper, and clearly $P_{2d}^{\text{sos}} \subseteq P_{2d}^{\text{pos}}$.

Univariate Polynomials

In fact, for univariate polynomials, nonnegative polynomials and sums of squares are the same thing!

Theorem

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Proof:

- Let $p \in P_{2d}^{\text{pos}}$. Then $p(x) = p_{2d} \prod_{i=1}^{2d} (x - a_i)$ for some $a_1, \ldots, a_{2d} \in \mathbb{C}$.

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- The complex roots come by conjugate pairs, and the real roots are of even multiplicity (because p does not change sign over $\mathbb{R}$).
- So after reordering the roots, $p(x) = p_{2d} \prod_{i=1}^{d} (x - a_i)(x - \overline{a_i}) = |q(z)|^2$,

  where $q(z) = \sqrt{p_{2d}} \prod_{i=1}^{d} (z - a_i) \in \mathbb{C}_d[z]$

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  where $q(z) = \sqrt{p_{2d}} \prod_{i=1}^{d} (z - a_i) \in \mathbb{C}_d[z]$
- Finally, $p = (\mathfrak{Re}\ q)^2 + (\mathfrak{Im}\ q)^2$.

Univariate Polynomials

- We have: $P_{2d}^{\text{sos}} = P_{2d}^{\text{pos}}$.
- And there is a nice characterization P_{2d}^{sos} of based on LMIs:

Theorem

Denote by $s_k(M)$ the sum of the kth antidiagonal of $M \in \mathbb{S}^d$:

$$s_k(M) = \sum_{\{0 \leq i,j \leq d\colon\ i+j=k\}} M_{ij}, \quad \forall k \in \{0, \ldots, 2d\}.$$

Then, $\boldsymbol{p} \in \mathbb{R}_d[x]$ is a sum of square iff

$$\exists M \succeq 0 : \quad s_k(M) = p_k, \forall k \in \{0, \ldots, 2d\}.$$

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Example:

$$p(x) = 5 - 14x + 2x^2 + 12x^3 + 4x^4 = [1,x,x^2] \overbrace{\begin{pmatrix} 5 & -7 & -4 \\ -7 & 10 & 6 \\ -4 & 6 & 4 \end{pmatrix}}^{M} [1,x,x^2]^T$$

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$M \succeq 0$, so we can decompose $M = HH^T$, where $H^T = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \end{pmatrix}$.

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So $p(x) = \|H[1, x, x^2]^T\|^2 = (1-x)^2 + (-2 + 3x + 2x^2)^2$.

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Corollary 1

We can check if $p \in \mathbb{R}_{2d}[x]$ is a nonnegative polynomial by using an SDP solver.

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Corollary 2

Let $p \in \mathbb{R}_{2d}[x]$. We can reformulate the problem $\inf_{x \in \mathbb{R}} p(x)$ as an SDP:

$$\inf_{x \in \mathbb{R}} p(x) = \sup \left\{ t : \ (p - t) \in P_{2d}^{sos} \right\}$$

Towards multivariate sum of squares

- Can we generalize this approach to multivariate polynomials $p \in \mathbb{R}_{2d}[x_1, \dots, x_n]$ of degree $2d$ in n variables ?
- Use multi-indices:

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \Delta(n,d)} p_{\boldsymbol{\alpha}}\, \boldsymbol{x}^{\boldsymbol{\alpha}}, \quad \text{where} \quad \boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

  where $\Delta(n, d) \subset \mathbb{Z}^n_{\geq 0}$ is the set of multi-indices with sum $\leq d$.
- Define cone of nonnegative polynomials and sum of squares of degree d on n variables, $P^{\text{pos}}_{n,2d}$ and $P^{\text{sos}}_{n,2d}$.

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- Define cone of nonnegative polynomials and sum of squares of degree d on n variables, $P^{\text{pos}}_{n,2d}$ and $P^{\text{sos}}_{n,2d}$.
- **Bad news:** $n = 1$ was a stroke of luck:

Theorem (Hilbert's theorem).

$$P^{\text{sos}}_{n,2d} = P^{\text{pos}}_{n,2d} \iff \Big((n = 1) \text{ or } (2d = 2) \text{ or } (n, 2d) = (2, 4) \Big).$$

Towards multivariate sum of squares

Example: Motzkin polynomial

$$p(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \in \mathbb{R}_6[x, y].$$

- p is nonnegative! (Arithmetic-Geometric mean inequality for x^2y^4, x^4y^2 and 1).

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- But we can show that p is not SOS. (Analytical proof, or, as in the case $n = 1$, we can test if $p \in P^{\text{sos}}_{2,6}$ by solving an SDP).

  Remark: The LMI for testing membership in $P^{\text{sos}}_{n,2d}$ involves a matrix of size $|\Delta(n, 2d)| = \binom{n+2d}{2d}$.

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  Remark: The LMI for testing membership in $P^{\text{sos}}_{n,2d}$ involves a matrix of size $|\Delta(n,2d)| = \binom{n+2d}{2d}$.

Good news:

$(1+x^2+y^2)\cdot p(x,y)$ **is a sum of square.**

$\rightarrow$ SDP-based certificate that the Motzkin polynomial is nonnegative.

Lasserre Hierarchy

For $r \geq 0$, define

$$P_{n,2d}^{(r)} = \left\{ p \in \mathbb{R}_{2d}[x_1, \ldots, x_n] : \quad x \mapsto (1 + \sum_{i=1}^{n} x_i^2)^r \cdot p(x) \text{ is SOS} \right\}.$$

The product of two SOS is SOS, hence:

$$P_{n,2d}^{\text{sos}} = P_{n,2d}^{(0)} \subseteq P_{n,2d}^{(1)} \subseteq P_{n,2d}^{(2)} \subseteq \ldots \subseteq P_{n,2d}^{\text{pos}}.$$

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- For a fixed r, checking whether a polynomial is in $P_{n,2d}^{(r)}$, or optimizing over $P_{n,2d}^{(r)}$, can be done in polytime; it's an SDP with a matrix variable $X \in \mathbb{S}_+^N$, where $N = \binom{n+2(d+r)}{2(d+r)}$.
- Moreover, the hierarchy converges [Putinar, 1993], and even for a finite r *generically* [Nie, 2014] (but which one?)

To go further...

- Possibility to handle *polynomially constrained* optimization problems: $\inf\{p_0(\boldsymbol{x}) : p_j(\boldsymbol{x}) \leq 0, \forall j\}$ for some polynomials $p_0, p_1, \ldots, p_m$.

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- Study of dual cone of $P_{n,2d}^{\text{pos}}$, which is the cone of moments

$$M_{n,2d} := \left\{ \left(\int_{\boldsymbol{x}\in\mathbb{R}^n} \boldsymbol{x}^{\boldsymbol{\alpha}} d\mu(\boldsymbol{x}) \right)_{\boldsymbol{\alpha}\in\Delta(n,2d)} : \mu \text{ nonnegative measure} \right\}.$$

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- Helton-Nie conjecture: *Is every convex semi-algebraic set semidefinite-representable?* $\rightarrow$ Refuted in 2017 by Scheiderer.

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- References

  - Laurent, M. (2009). Sums of squares, moment matrices and optimization over polynomials. In *Emerging applications of algebraic geometry*, pp. 157–270.
  - Lasserre, J. B. (2010). Moments, positive polynomials and their applications (Vol. 1). World Scientific.

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Lift-and-project representations

Recall the power of lift-and-project representations: The polytope $B_1 = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$ can written as:

- The intersection of 2^n halfspaces

$$B_1 = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{h}^T\boldsymbol{x} \leq 1, \forall h \in \{-1, 1\}^n\}.$$

- The projection of a $2n$-dimensional polytope defined by $2n + 1$ facets.

$$B_1 = \{\boldsymbol{x} \in \mathbb{R}^n : \ \exists \boldsymbol{u} \in \mathbb{R}^n, -\boldsymbol{u} \leq \boldsymbol{x} \leq \boldsymbol{u}, \ \mathbf{1}^T\boldsymbol{u} \leq 1\}.$$

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Natural question

Given a polytope P, what is the smallest number k such that P is the projection of a polytope with k facets?

- Question has important application for combinatorial optimization (polytopes with vertices in $\{0, 1\}^n$)

Slack matrix

Consider a polytope P with vertices $\boldsymbol{v}_j$ ($\forall j \in [N]$) and facets $h_i(\boldsymbol{x}) := g_i - \boldsymbol{f}_i^T \boldsymbol{x} \geq 0$ ($\forall i \in [M]$), so that

$$P = \textbf{conv}\,\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} = \{\boldsymbol{x} \in \mathbb{R}^n : F\boldsymbol{x} \leq \boldsymbol{g}\}.$$

Definition (Slack matrix)

The *slack matrix* of P is the matrix $S \in \mathbb{R}^{M \times N}$ such that

$$S_{ij} := h_i(\boldsymbol{v}_j)$$

Example:
cf. blackboard

Yannakakis' theorem

Natural question

Given a polytope P, what is the smallest number k such that P is the projection of a polytope with k facets?

- Formally, we are looking for the smallest $k \in \mathbb{N}$ such that for some vectors $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\boldsymbol{c} \in \mathbb{R}^k$ and matrices Π, L of appropriate dimensions, it holds

$$P = \{\Pi \boldsymbol{y} + \boldsymbol{x}_0 | \quad L\boldsymbol{y} \preceq_{\mathbb{R}^k_+} \boldsymbol{c}\}.$$

- This k is called the *extension complexity* of P, $\mathbf{xc}(P) := k$.

Theorem (Yannakakis, 1991).

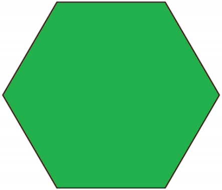
$\mathbf{xc}(P)$ is equal to the *nonnegative rank* of the slack matrix S, that is, the smallest k such that a factorization $S = AB$ exists, with $A \in \mathbb{R}_+^{M \times k}$, $B \in \mathbb{R}_+^{k \times N}$.

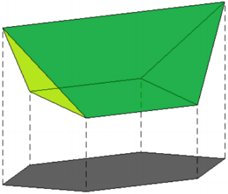
Example: The regular hexagon

The slack matrix of the regular hexagon admits a nonnegative factorization of rank 5:

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The hexagon is the projection of a polytope with 5 facets:





K-lift

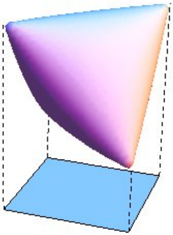
We are going to prove a generalization of Yannakakis' result for proper cones, due to Gouveia, Parrilo and Thomas (2013).

Definition (K-lift)

Let K be a proper cone. We say that P has a K-lift if

$$P = \{\Pi \boldsymbol{y} + \boldsymbol{x}_0 | \quad L\boldsymbol{y} \preceq_K \boldsymbol{c}\}$$

for some vectors $\boldsymbol{x}_0$, $\boldsymbol{c}$ and matrices Π, L.



Example:
The square $[-1,1]^2$ has an $\mathbb{S}_+^3$-lift:

$$P = [-1,1]^2 = \left\{ (x,y) : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$$

Conic Extension Complexity

Definition (Closed cone family)

An ordered family of proper cones $\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ is called *closed* if for every face F of K_i, F is isomorphic to K_j for some $j \leq i$.

Example

- $\mathcal{K} = (\mathbb{R}^i_+)_{i \in \mathbb{N}}, \quad \mathcal{K} = (\mathbb{S}^i_+)_{i \in \mathbb{N}}$
- We can also define closed families of products of Lorrentz cones

Given a closed cone family $\mathcal{K}$, what is the smallest i such that P has a K_i-lift ? This is the $\mathcal{K}$*-extension complexity* of P, $\mathbf{xc}_{\mathcal{K}}(P)$.

For $\mathcal{K} = (\mathbb{R}^i_+)_{i \in \mathbb{N}}$, this is exactly the setting of the Yannakakis theorem: for which i can P be written as the projection of a polytope with i facets?

Cone-rank

Given a closed cone family $\mathcal{K}$, what is the smallest i such that P has a K_i-lift ?

We will prove that this number is equal to the $\mathcal{K}$-rank of the slack matrix S:

Definition (Cone rank).

Let $\mathcal{K} = (K_k)_{k\in\mathbb{N}}$ be a closed cone family. We define the $\mathcal{K}$-rank of $S \in \mathbb{R}^{m\times n}$ as the smallest $k \in \mathbb{N}$ such that there exists $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \succeq_{K_k^*} \mathbf{0}$ and $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n \succeq_{K_k} \mathbf{0}$ such that

$$S_{ij} = \langle \boldsymbol{a}_i, \boldsymbol{b}_j \rangle, \quad \forall (i,j) \in [m] \times [n].$$

That is, we have $S = AB$ for some matrix A with rows in K_k^* and some matrix B with columns in K_k.

The conic Yannakakis theorem

Let P be a polytope and S its slack matrix.

Theorem [Gouveia, Parrilo & Thomas 2013]

For any closed cone family $\mathcal{K}$, it holds $\mathbf{xc}_{\mathcal{K}}(P) = \mathbf{rank}_{\mathcal{K}}(S)$.

Remark: This shows the equivalence between a hard geometric problem and a hard algebraic problem...

The conic Yannakakis theorem

Proof sketch: Let $K \in \mathcal{K}$ be *minimal*, s.t. a K-lift exists:

$$P = \{\Pi \boldsymbol{y} + \boldsymbol{x}_0 | \quad L\boldsymbol{y} \preceq_K \boldsymbol{c}\}.$$

We will construct a K-factorization of S.

- Consider a facet $h_i(\boldsymbol{x}) = g_i - f_i^T \boldsymbol{x} \geq 0$ of P. We have

$$\sup_{\boldsymbol{x} \in P} f_i^T \boldsymbol{x} = g_i$$

$$\text{[}P\text{ has a }K\text{-lift]} \iff \sup_{\boldsymbol{y}} \{f_i^T (\Pi \boldsymbol{y} + \boldsymbol{x}_0) : L\boldsymbol{y} \preceq_K \boldsymbol{c}\} = g_i$$

$$\text{[strong duality]} \iff \inf_{\boldsymbol{z}} \{\boldsymbol{z}^T \boldsymbol{c} - f_i^T \boldsymbol{x}_0 : \Pi^T \boldsymbol{f}_i = L^T \boldsymbol{z},\ \boldsymbol{z} \succeq_{K^*} \boldsymbol{0}\} = g_i$$

It can be shown that the primal problem is strictly feasible, as otherwise this contradicts that K is the *minimal cone* such that a K-lift exists. Hence, the dual is solvable, and

$$\exists \boldsymbol{z}_i \succeq_{K^*} \boldsymbol{0} : \Pi^T \boldsymbol{f}_i = L^T \boldsymbol{z}_i,\ \boldsymbol{z}_i^T \boldsymbol{c} - f_i^T \boldsymbol{x}_0 = g_i.$$

The conic Yannakakis theorem

- For all facets $h_i(\boldsymbol{x}) = g_i - f_i^T \boldsymbol{x} \geq 0$,

$$\exists \boldsymbol{a}_i \succeq_{K^*} \boldsymbol{0} : \quad \boldsymbol{f}_i^T \Pi = \boldsymbol{a}_i^T L, \quad \boldsymbol{a}_i^T \boldsymbol{c} - f_i^T \boldsymbol{x}_0 = g_i.$$

- For all vertex $\boldsymbol{v}_j$ of P, define $\boldsymbol{b}_j := \boldsymbol{c} - L\boldsymbol{y}_j \succeq_K \boldsymbol{0}$ for some arbitrary vector $\boldsymbol{y}_j$ s.t. $\boldsymbol{v}_j = \Pi \boldsymbol{y}_j + \boldsymbol{x}_0$.

Now, we evaluate $S_{ij} = h_i(\boldsymbol{v}_j)$ for some $(i, j) \in [M] \times [N]$:

$$\begin{aligned} S_{ij} = g_i - \boldsymbol{f}_i^T \boldsymbol{v}_j &= \boldsymbol{a}_i^T \boldsymbol{c} - f_i^T \boldsymbol{x}_0 - \boldsymbol{f}_i^T (\Pi \boldsymbol{y}_j + \boldsymbol{x}_0) \\ &= \boldsymbol{a}_i^T \boldsymbol{c} - \boldsymbol{f}_i^T \Pi \boldsymbol{y}_j \\ &= \boldsymbol{a}_i^T \boldsymbol{c} - \boldsymbol{a}_i^T L \boldsymbol{y}_j \\ &= \boldsymbol{a}_i^T \boldsymbol{b}_j. \end{aligned}$$

Since $\boldsymbol{a}_i \succeq_{K^*} \boldsymbol{0}$ and $\boldsymbol{b}_j \succeq_K \boldsymbol{0}$, this is a K-factorization of S.

The conic Yannakakis theorem

Conversely, assume that the slack matrix has a K-factorization: $\exists \boldsymbol{a}_1, \ldots, \boldsymbol{a}_N \succeq_{K^*} \mathbf{0},\ \boldsymbol{b}_1, \ldots, \boldsymbol{b}_M \succeq_K \mathbf{0}$:

$$g_i - \boldsymbol{f}_i^T \boldsymbol{v}_j = \langle \boldsymbol{a}_i, \boldsymbol{b}_j \rangle, \quad \forall (i,j) \in [M] \times [N].$$

We will show that P has a $K-$lift.

Claim: $P = \{\boldsymbol{x} |\ \exists \boldsymbol{b} \succeq_K \mathbf{0} :\ \forall i, \quad g_i - \boldsymbol{f}_i^T \boldsymbol{x} = \langle \boldsymbol{a}_i, \boldsymbol{b} \rangle\}$ (1)

- $\supseteq$: For all facets, it holds $h_i(\boldsymbol{x}) = \langle \boldsymbol{a}_i, \boldsymbol{b} \rangle \geq 0$. So $\boldsymbol{x} \in P$.
- $\subseteq$: Let $\boldsymbol{x} = \sum_j \lambda_j \boldsymbol{v}_j \in P$, with $\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^T \boldsymbol{\lambda} = 1$. Define

  $\boldsymbol{b} = \sum_j \lambda_j \boldsymbol{b}_j \succeq_K \mathbf{0}$. Then, for all facets $i \in [M]$,

$$g_i - \boldsymbol{f}_i^T \boldsymbol{x} = \sum_j \lambda_j (g_i - \boldsymbol{f}_i^T \boldsymbol{v}_j) = \sum_j \lambda_j \langle \boldsymbol{a}_i, \boldsymbol{b}_j \rangle = \langle \boldsymbol{a}_i, \boldsymbol{b} \rangle.$$

Then, it is standard algebra to show that (1) is a $K-$lift of P.

To go further...

- Exponential lower bounds on the (conic) extension complexity of certain polytopes
- Study of the ratio $\dfrac{\mathbf{xc}_{(\mathbb{S}_+^k)}(P_d)}{\mathbf{xc}_{(\mathbb{R}_+^k)}(P_d)}$ for some families of polytopes $(P_d)_{d\in\mathbb{N}}$.
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