Exercise Sheet 5 (due date for Exercises 5.1 - 5.3: Jan. 17)

Exercise 5.1 (Homework)

In the lecture, we studied the SDP relaxation of MAXCUT

$$\begin{aligned} \max_X \quad & \frac{1}{4}\langle W, J - X\rangle \\ s.t. \quad & \operatorname{diag}(X) = \mathbf{1} \\ & X \succeq 0. \end{aligned}$$

We will now study an alternative SDP relaxation of MAXCUT. This relaxation relies on the Laplacian matrix of G, defined as

$$L = \sum_{ij \in E} w_{ij} (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^T.$$

1. Let $\boldsymbol{x}$ be a vector in $\{-1, 1\}^n$. Show that

$$\frac{1}{2} \sum_{ij \in E} w_{ij}(1 - x_i x_j) = \frac{1}{4} \sum_{1 \leq i,j \leq n} L_{ij} x_i x_j,$$

   and conclude that the following SDP is a relaxation of MAXCUT:

$$\begin{aligned} p^* = \max_X \quad & \frac{1}{4}\langle L, X\rangle \\ s.t. \quad & \operatorname{diag}(X) = \mathbf{1} \\ & X \succeq 0 \end{aligned}$$

2. Show that the dual of this SDP is

$$\begin{aligned} d^* = \min_{\boldsymbol{y}} \quad & \frac{1}{4}\, \mathbf{1}^T \boldsymbol{y} \\ s.t. \quad & L \preceq \operatorname{Diag}(\boldsymbol{y}) \end{aligned}$$

3. Does strong duality hold?

4. Show that

$$\operatorname{maxcut}(G) \leq \frac{n}{4} \lambda_{\max}(L).$$

Exercise 5.2 (Homework)

Copositive programming formulation of the maximum independent set

A conic optimization problem involving the cone $\mathcal{C}_n$ (or its dual) is called a *copositive program*. The goal of this exercise is to show that the maximum independent set of a graph can be computed by solving a copositive program. Hence copositive programming is intractable, but it is known that copositive programs can be *approximated* by SDPs. Let $G = (V, E)$ be a simple graph with n vertices. We recall that the stability number satisfies $\alpha(G) \leq \vartheta(G)$, where $\vartheta(G)$ is the Lovasz-theta number of G:

$$\begin{aligned} \vartheta(G) = \max_X \quad & \langle J, X \rangle \\ s.t. \quad & \langle I, X \rangle = 1 \\ & X_{ij} = 0, \qquad \forall ij \in E \\ & X \succeq_{\mathbb{S}_+^n} 0. \end{aligned}$$

In what follows, we will show that $\alpha(G) = \vartheta^*(G)$, where $\vartheta^*(G)$ is the value of the copositive program obtained by replacing the constraint "$X \succeq_{\mathbb{S}_+^n} 0$" by "$X \succeq_{\mathcal{C}_n^*} 0$".
Recall the following definitions: $\boldsymbol{x} \in K$ is an *extreme ray* of a convex cone K if the only possibility to express $\boldsymbol{x}$ as a barycenter of two other rays $\boldsymbol{y}, \boldsymbol{z} \in K$ is to take $\boldsymbol{y} = \alpha \boldsymbol{x}$ and $\boldsymbol{z} = \beta \boldsymbol{x}$ for some scalars α and β. Similarly, $\boldsymbol{x} \in S$ is an *extreme point* of a convex set S if the only possibility to express $\boldsymbol{x}$ as a barycenter of two other points $\boldsymbol{y}, \boldsymbol{z} \in S$ is to take $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}$.
You can use (without proof) the following result:

Let K be a convex cone, and let $H = \{\boldsymbol{x} : \boldsymbol{a}^T \boldsymbol{x} = b\}$ be an hyperplane, with $\boldsymbol{a} \in \mathbf{int} K^*$. Then it holds

$$\textbf{ext-points}(K \cap H) = \textbf{ext-rays}(K) \cap H.$$

1. Define $\mathcal{K} = \{X \in \mathcal{C}_n^* : \quad \forall ij \in E, X_{ij} = 0\}$, and observe that $\mathcal{K}$ is a cone. Show that X is an extreme ray of $\mathcal{K}$ iff $X = \boldsymbol{x}\boldsymbol{x}^T$ for some $\boldsymbol{x} \in \mathbb{R}_+^n$ supported by a stable set of G (i.e., $S = \{i : x_i \neq 0\}$ is stable).
2. Use the result of 1. to identify the set of extreme points of the feasible set of the copositive program for $\vartheta^*(G)$. Conclude that $\vartheta^*(G) = \alpha(G)$.

Exercise 5.3 (Homework)

The partition problem is defined as follows: Given integers $a_1, \ldots, a_n$, does there exist a subset $S \subseteq [n]$ such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$?

An optimization version of this problem is the following:

$$\underset{S \subseteq [n]}{\textbf{minimize}} \left(\sum_{i \in S} a_i - \sum_{i \notin S} a_i \right)^2$$

1. Reformulate the problem as a binary quadratic program and formulate an SDP relaxation.
2. Show how to change the relaxation to handle the constraint $|S| = k$.