# Exercise Sheet 4 (due date for Exercises 4.1 - 4.3: Dec. 13)

## Exercise 4.1 (Homework)

*Lagrangian dual of a SDP.*

1. Show that the Lagrangian dual of the following SDP

$$\begin{aligned} p^* = \max \quad & \langle A, X \rangle \\ s.t. \quad & \mathbf{tr}\; X = 1 \\ & X \succeq 0. \end{aligned}$$

   is equivalent to

$$\begin{aligned} d^* = \min_{\lambda \in \mathbb{R}} \quad & \lambda \\ s.t. \quad & \lambda I \succeq A. \end{aligned}$$

2. What does the weak duality theorem tell us about $p^*$ and $d^*$?
3. Give a closed-form expression for the value of $d^*$.
4. How can the strong duality theorem be applied?
5. Find a feasible $X$ such that $\langle A, X \rangle = d^*$.
   (*Hint:* You can search for a matrix $X$ of the form $X = \boldsymbol{x}\boldsymbol{x}^T$ for some well chosen vector $\boldsymbol{x} \in \mathbb{R}^n$.)

## Exercise 4.2 (Homework)

*An exact penalty method for inequality constraints.* Consider the problem

$$\begin{aligned} p^* = \min_{\boldsymbol{x} \in \mathbb{R}^n} \quad & f_0(\boldsymbol{x}) \\ s.t. \quad & f_i(\boldsymbol{x}) \leq 0, \quad i = 1, \ldots, m, \end{aligned} \tag{P}$$

where the functions $f_i$ are convex. In an exact penalty method, we solve the (unconstrained) auxiliary problem

$$\min_{\boldsymbol{x}} \quad \Phi(\boldsymbol{x}) := f_0(\boldsymbol{x}) + \alpha \max_{i=1,\ldots,m} \max(0, f_i(\boldsymbol{x})) \tag{$P_\alpha$}$$

where $\alpha > 0$ is a parameter. The second term in $\Phi$ penalizes deviations of $x$ from feasibility.

1. Show that $\Phi$ is convex.
2. The auxiliary problem can be rewritten as

$$\begin{aligned} \min_{\boldsymbol{x}, y} \quad & \Phi(\boldsymbol{x}) := f_0(\boldsymbol{x}) + \alpha y \\ s.t. \quad & f_i(\boldsymbol{x}) \leq y, \quad (i = 1, \ldots, m) \\ & 0 \leq y, \end{aligned} \tag{$P'_\alpha$}$$

   where the variables are $\boldsymbol{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

Show that the Lagrangian dual of $(P'_\alpha)$ simplifies to

$$\sup\{g(\boldsymbol{\lambda}) :\ \boldsymbol{\lambda} \geq \mathbf{0},\ \sum_{i=1}^{m} \lambda_i \leq \alpha\},$$

where $g$ is the Lagrange dual function of the original problem $(P)$.

3. We know that if Problem $(P)$ is *strictly feasible* and *bounded*, then there exists a vector $\boldsymbol{\lambda}^* \in \mathbb{R}^m_+$ that solves the dual problem and strong duality holds: $p^* = d^* = g(\boldsymbol{\lambda}^*)$. Show that, if we have in addition $\alpha > \sum_i \lambda_i^*$, then any optimal solution of Problem $(P_\alpha)$ is also an optimal solution of $(P)$.

## Exercise 4.3 (Homework)

*Binary least squares.* We consider the non-convex least-squares approximation problem with binary constraints

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 \\ s.t. \quad & x_k \in \{-1, 1\}, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. We assume that $\mathbf{rank} A = n$, i.e., $A^T A$ is nonsingular.

1. Derive the Lagrangian dual of this problem, when each constraint $x_k \in \{-1, 1\}$ is replaced by $x_k^2 = 1$. You can ignore the case in which $A^T A + diag(\boldsymbol{\mu})$ is positive semidefinite but singular.

2. Reformulate the Lagrangian dual as an SDP. Can we assert that strong duality holds for this problem? (You can assume that the SDP you derived is still valid in the singular case.)

## Exercise 4.4

*An SDP where strong duality fails.* Consider the problem

$$\begin{aligned} \min \quad & x_2 \\ s.t. \quad & \begin{pmatrix} x_2 + 1 & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & x_2 & 0 \end{pmatrix} \succeq 0. \end{aligned}$$

1. Derive the Lagrangian dual of this problem.

2. Compute $p^*$ and $d^*$.