Exercise Sheet 3 (due date for homework exercises 3.1 to 3.3: Nov. 29)

Exercise 3.1 (Homework)

For a non-square matrix $Y \in \mathbb{R}^{m\times k}$, the norm of Y induced by the Euclidean norm is the spectral norm, defined by $\|Y\| = \sqrt{\lambda_{\max}(YY^T)}$.

1. Show that $\|Y\| \leq t \iff \begin{bmatrix} t\,I_m & Y \\ Y^T & t\,I_k \end{bmatrix} \succeq 0$

Now, let P be the polyhedron $\{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\}$ and define $F(\boldsymbol{x}) = F_0 + x_1F_1 + x_2F_2 + \ldots + x_nF_n$, and $G(\boldsymbol{x}) = G_0 + x_1G_1 + x_2G_2 + \ldots + x_nG_n$, where for all i, $F_i \in \mathbb{R}^{m\times k}$ and $G_i \in \mathbb{S}^m$. Show how to pose the following problems as SDPs (You don't need to put the SDP in a standard form):

2. **minimize** $\|F(\boldsymbol{x})\|$ over $\boldsymbol{x} \in P$.
3. **minimize** $\lambda_{\max}(G(\boldsymbol{x})) - \lambda_{\min}(G(\boldsymbol{x}))$ over $\boldsymbol{x} \in P$.
4. **minimize** $\sum_{i=1}^m |\lambda_i(\boldsymbol{x})|$ over $\boldsymbol{x} \in P$, where $\lambda_1(\boldsymbol{x}) \geq \cdots \geq \lambda_m(\boldsymbol{x})$ are the eigenvalues (counted with multiplicity) of $G(\boldsymbol{x})$.
   Hint: You can use the following result: If G is a symmetric matrix, it admits a decomposition of the form $G = G_+ - G_-$, where $G_+ \succeq 0$ and $G_- \succeq 0$, and which satisfies the following properties:
   (i) The nonzero eigenvalues of G^+ are the positive eigenvalues of G;
   (ii) The nonzero eigenvalues of G^- are the absolute values of the negative eigenvalues of G;
   (iii) If $G = C - D$ for another decomposition with $C \succeq 0, D \succeq 0$, then we have $\operatorname{trace} C \geq \operatorname{trace} G_+$, and $\operatorname{trace} D \geq \operatorname{trace} G_-$.

Exercise 3.2 (Homework)

We consider the SDP

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^3}{\textbf{minimize}} \quad & x_1 - 2x_3 \\ & x_1 + x_2 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \succeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Show how to reformulate this SDP under the *standard form*: $\min\langle C, X\rangle,\ s.t.\ \langle A_i, X\rangle = b_i,\ X \succeq 0$. (You do not have to give the complete SDP, but at least an example for each type of constraint of your SDP).

Exercise 3.3 (Homework)

Show that the following functions have a $K_{\exp}$-representation, where $K_{\exp}$ denotes the exponential cone:

1. The relative entropy (Kullback-Leibler divergence) of two non-negative vectors $\boldsymbol{x}, \boldsymbol{y} \geq 0$:

   $$f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^n x_i \log \frac{x_i}{y_i},$$

   with the convention discussed in the lecture, when the variables attain the value 0.
2. The log-sum-exp function: $g(\boldsymbol{x}) = \log(\sum_{i=1}^n e^{x_i})$

Exercise 3.4

Formulate the problem of maximizing $f(\boldsymbol{x})$ over the polyhedron $P = \{\boldsymbol{x} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ (we assume that P is nonempty) as an SOCP, where

1. $f(\boldsymbol{x}) = n(\sum_i \frac{1}{x_i})^{-1}$ is the harmonic mean of $\boldsymbol{x}$. To do so, show the following:
   (a) Let $t \geq 0$. We have $f(\boldsymbol{x}) \geq t$ if and only if there exists $\boldsymbol{\mu} \geq 0$ s.t.
   $$\begin{cases} t^2 \leq \mu_i x_i n \quad \forall i = 1, \dots, n \\ t \geq \sum_{i=1}^n \mu_i. \end{cases}$$
   (b) Conclude.
2. $f(\boldsymbol{x}) = (\prod_i x_i)^{\frac{1}{n}}$ is the geometric mean of $\boldsymbol{x}$. To do so, show the following:
   (a) Let $t \geq 0$. We have $(x_1 x_2 x_3 x_4)^{\frac{1}{4}} \geq t$ if and only if $\exists u, v \geq 0$ s.t.
   $$\begin{cases} u^2 \leq x_1 x_2 \\ v^2 \leq x_3 x_4 \\ t^2 \leq uv. \end{cases}$$
   (b) Explain briefly how (a) can be extended to any n being a power of two.
   (c) How can this be extended to any positive integer n?