Exercise Sheet 2 (due date for homework exercises 2.1 to 2.3: Nov. 15)

Exercise 2.1 (Homework)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, with $\mathbf{dom} f = \mathbb{R}^n$. Moreover, let f be bounded from above on $\mathbb{R}^n$. Show that f is constant.

Exercise 2.2 (Homework)

We consider an optimization problem of the form

$$\underset{\boldsymbol{x}\in\Delta_n^=}{\mathbf{minimize}} \quad f(\boldsymbol{x}) \qquad \text{with } \mathbf{dom} f \supset \Delta_n^=,$$

where f is convex and differentiable and $\Delta_n^= = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \mathbf{0}, \sum_{i=1}^n x_i = 1\}$ is the probability simplex of $\mathbb{R}^n$. The first order optimality conditions for this problem tell us that $\boldsymbol{x} \in \Delta_n^=$ is optimal iff

$$\forall \boldsymbol{y} \in \Delta_n^=, \quad \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) \geq 0. \tag{1}$$

1. Show that $\Delta_n^=$ is the convex hull of the vectors $\{\boldsymbol{e}_i | i = 1, \ldots, n\}$, where $\boldsymbol{e}_i$ is the ith standard unit vector of $\mathbb{R}^n$, i.e., $\boldsymbol{e}_i \in \mathbb{R}^n$ is the vector with entry 1 in the ith component, and 0's everywhere else.

2. Use this to show that condition (1) is equivalent to

$$\nabla f(\boldsymbol{x})^T \boldsymbol{x} \leq \min_{i=1,\ldots,n} \frac{\partial f(\boldsymbol{x})}{\partial x_i}.$$

3. Derive the following optimality conditions over the probability simplex: $\boldsymbol{x} \in \Delta_n^=$ is optimal if and only if there exists a $\lambda \in \mathbb{R}$ such that

$$\forall i = 1, \ldots, n, \quad \begin{cases} x_i \geq 0 \text{ and } \frac{\partial f(\boldsymbol{x})}{\partial x_i} = \lambda \\ \quad\quad\text{or} \\ x_i = 0 \text{ and } \frac{\partial f(\boldsymbol{x})}{\partial x_i} > \lambda. \end{cases}$$

Exercise 2.3 (Homework)

Show that the separation problem can be solved for the non-negative orthant $\mathbb{R}^n_+$, the Lorentz cone $\mathbb{L}^n_+$ and the positive semidefinite cone $\mathbb{S}^n_+$, i.e., either assert that an element x is in the cone K or explicitly give a hyperplane separating x and K, that is, find $h \neq 0$ such that

$$\langle h, y \rangle \geq \langle h, x \rangle \quad \forall y \in K.$$

Reminder: $\mathbb{L}^n_+ := \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|x\|_2 \leq t\}$

Exercise 2.4

Let P be an optimization problem of the form

$$p^* = \inf_{\boldsymbol{x} \in C} f(x),$$

where C is convex and the function f is convex over C.

1. Show that the optimal set, i.e. the set of optimal solutions $X_* = \{\boldsymbol{x} \in C : \ f(\boldsymbol{x}) = p^*\}$ is convex.
2. Show that the set of $\epsilon-$suboptimal solutions $X_\epsilon = \{\boldsymbol{x} \in C : \ f(\boldsymbol{x}) \leq p^* + \epsilon\}$ is convex.
3. Show that if f is *strictly* convex, then X^* contains at most one point.

Exercise 2.5

Consider the optimization problem

$$\begin{aligned} \textbf{minimize} \quad & f_0(x_1, x_2) \\ s.t. \quad & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value:

1. $f_0(x_1, x_2) = x_1 + x_2$
2. $f_0(x_1, x_2) = -x_1 - x_2$
3. $f_0(x_1, x_2) = x_1$
4. $f_0(x_1, x_2) = \max(x_1, x_2)$
5. $f_0(x_1, x_2) = x_1^2 + 9x_2^2$