# Exercise Sheet 1 (due date for homework exercises 1.1 to 1.3: Nov. 01)

## Exercise 1.1 (Homework)

Let $A \in \mathbb{S}^n_{++}$ and $B \in \mathbb{S}^n$.

1. Show that for all matrices $U$ of appropriate size, $A \preceq B \implies U^T AU \preceq U^T BU$.
2. Show that $A \preceq \boldsymbol{I}_n \iff A^{-1} \succeq \boldsymbol{I}_n$. (Hint: Use the matrix $A^{\frac{1}{2}}$)
3. Show that $A \preceq B$ implies that $B$ is invertible, and $A^{-1} \succeq B^{-1}$.

   We define the ellipsoid $\mathcal{E}_A := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T A^{-1}\boldsymbol{x} \leq 1\}$.
4. Let now $B$ be positive definite. Conclude that $A \preceq B \iff \mathcal{E}_A \subseteq \mathcal{E}_B$.

## Exercise 1.2 (Homework)

1. Let $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{n\times n}$ invertible. We define

   $$S_1 := \{A_1 x : x \geq 0\} \quad \text{and} \quad S_2 := \{x : A_2 x \geq 0\}.$$

   Determine the dual cones of $S_1$ and $S_2$ and give it in the form

   $$\{y : By \geq 0\}.$$

   for some suitable $B$.
2. Consider $K := \{x : x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}$. Determine the dual cone of $K$.

## Exercise 1.3 (Homework)

Let $C \subseteq \mathbb{R}^n$ with nonempty interior. Moreover, let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, and $g : \mathbb{R}^n \to \mathbb{R}$ be concave, with $\mathbf{dom} f = \mathbf{dom}\, g = C$. Suppose that the inequality $g(\boldsymbol{x}) \leq f(\boldsymbol{x})$ holds everywhere on $C$. Show that we can fit an affine function between $f$ and $g$, i.e., there exists an affine function $h$ such that

$$\forall \boldsymbol{x} \in C, \quad g(\boldsymbol{x}) \leq h(\boldsymbol{x}) \leq f(\boldsymbol{x}).$$

## Exercise 1.4

We define the *support function* of a set $C \subseteq \mathbb{R}^n$, $C \neq \emptyset$, as

$$S_C(\boldsymbol{x}) = \sup\{\boldsymbol{x}^T \boldsymbol{y} | \boldsymbol{y} \in C\},$$

with $\mathbf{dom}\, S_C = \{\boldsymbol{x} \in \mathbb{R}^n : \sup_{\boldsymbol{y}\in C} \boldsymbol{x}^T \boldsymbol{y} < \infty\}$.

1. Show that $S_C$ is a convex function
2. Show that $S_C = S_{\mathbf{conv}\, C}$
3. Show that $S_{A+B} = S_A + S_B$
4. Show that $S_{A\cup B} = \max\{S_A, S_B\}$
5. Let $B$ be closed and convex. Show that $A \subseteq B \iff \forall \boldsymbol{x} : S_A(\boldsymbol{x}) \leq S_B(\boldsymbol{x})$