# Chapter VIII: SDP in Combinatorial Optimization

## 1 Stable Sets and Graph Coloring

In this section, we consider a simple, undirected graphs $G = (V, E)$. Formally, an edge between the edges $u \in V$ and $v \in V$ should be represented by an unordered pair $\{u, v\} \in E$, but we will write $uv \in E$ for the sake of simplicity. In particular, $uv \in E \iff vu \in E$.

**Definition 1** (Stable set)**.** Let $G$ be a simple graph. A subset of vertices $S \subseteq V$ is called *stable* (or *independent*) if

$$u \in S, v \in S \implies uv \notin E.$$

The stable set problem is defined as follows: *Given a simple graph $G = (V, E)$, find the largest stable set of $G$.* The *stability number* (or *independent number*) of the graph is defined as the cardinality of the largest stable set. We denote it by $\alpha(G)$.

The stable set problem admits the following formulation as an integer linear program:

$$\begin{aligned} \alpha(G) = \max_{\boldsymbol{x}} \quad & \sum_{v \in V} x_v \\ s.t. \quad & x_u + x_v \leq 1, \quad \forall uv \in E \\ & \boldsymbol{x} \in \{0,1\}^V \end{aligned} \tag{1}$$

The fractional relaxation of the above IP, where the constraints $x_v \in \{0, 1\}$ are replaced by $0 \leq x_v \leq 1$, can be shown to be half-integer (there always exists an optimal solution such that $x_v \in \{0, \frac{1}{2}, 1\}$, $\forall v \in V$). These half-integer solutions can be quite poor; for example, it is easy to see that for the complete graph $K_n$, the fractional solution of (1) is $x_v = \frac{1}{2}$ for all $v \in V$, which yields the fractional value of $\sum_{v \in V} x_v = \frac{n}{2}$, while $\alpha(K_n) = 1$.

We will see that semidefinite programming allows us to formulate a stronger relaxation, which even yields an exact algorithm for a large class of graphs.

**Definition 2** (Clique number)**.** A clique of $G$ is a subset $U \subseteq V$ such that $u, v \in U \implies uv \in E$. The largest cardinality of a clique is called the clique number, and is denoted by $\omega(G)$.

**Definition 3** (Chromatic number)**.** A $k$-coloring of $G$ is a partition of its vertices in exactly $k$ stable sets. The chromatic number of $G$ is the smallest number $k$ such that a $k-$coloring exists, and is denoted by $\chi(G)$

**Definition 4** (Clique cover number)**.** A $k$-clique cover of $G$ is a partition of its vertices in exactly $k$ cliques. The clique cover number of $G$ is the smallest number $k$ such that a $k-$clique cover exists, and is denoted by $\bar{\chi}(G)$.

To summarize,

$$\begin{aligned}
\alpha(G) &= \max\{|S| : S \text{ is a stable set of } G\} && \text{[stability number]}\\
\omega(G) &= \max\{|C| : C \text{ is a clique of } G\} && \text{[clique number]}\\
\chi(G) &= \min\{k : V = S_1 \uplus \cdots \uplus S_k, \text{where the } S_i\text{'s are stable sets of } G\} && \text{[chromatic number]}\\
\bar{\chi}(G) &= \min\{k : V = C_1 \uplus \cdots \uplus C_k, \text{where the } C_i\text{'s are cliques of } G\}. && \text{[clique cover number]}
\end{aligned}$$

Let $\bar{G}$ denote the complementary graph of $G$, that is, $\bar{G} = (V, \bar{E})$, where $\bar{E} = E_K \setminus E$ and $E_K$ is the set of edges on the complete graph over $V$. In other words, $i$ and $j$ are connected in $\bar{G}$ iff they are not connected in $G$. It is straigtforward to show that

**Proposition 1.** *$\alpha(G) = \omega(\bar{G})$ and $\chi(G) = \bar{\chi}(\bar{G})$.*

*Proof.* This results from the simple observation

$$S \text{ is stable in } G \iff S \text{ is a clique in } \bar{G}.$$

□

It is also clear that

$$\omega(G) \leq \chi(G),$$

because each vertex of a clique must be assigned to different colors. Hence, by taking the complementary graph, we obtain

$$\alpha(G) \leq \bar{\chi}(G).$$

A graph in which the equality $\omega(G') = \chi(G')$ holds for all induced subgraphs $G'$ of $G$ (including $G' = G$) is called a *perfect graph*. Lovász showed in 1972 that a graph is perfect iff its complementary is perfect. Hence, in a perfect graph it also holds that $\alpha(G) = \bar{\chi}(G)$. The class of perfect graphs was finally characterized in 2006 by Chudnovsky, Robertson, Seymour, and Thomas, in a theorem that we will not prove:

**Theorem 2.** *A graph $G$ is perfect if and only if neither $G$ nor $\bar{G}$ contains an odd cycle of length $\geq 5$ as an induced subgraph.*

We also point out that there is a polynomial-time algorithm that recognizes whether a graph is perfect.

We will now present what is often refered as the Sandwich theorem of Lovász. It states that there is a function of $G$, equal to the value of some SDP, that lies between $\alpha(G)$ and $\bar{\chi}(G)$. As a consequence, $\alpha(G)$ and $\bar{\chi}(G)$ can be computed in polynomial time if $G$ is perfect. In contrast, note that computing any of $\alpha(G)$ or $\chi(G)$ is NP-hard for general graphs.

**Definition 5.** The theta-function of Lovász is defined for all graphs $G$, as follows:

$$\begin{aligned} \vartheta(G) = \max_{X \in \mathbb{S}^n} \quad & \langle J, X\rangle \\ s.t. \quad & \langle I, X\rangle = 1 \\ & X_{ij} = 0, \quad \forall ij \in E \\ & X \succeq 0. \end{aligned} \tag{2}$$

Recall that $I$ denotes the identity matrix and $J = \mathbf{1}\mathbf{1}^T$ is the matrix of all-ones. Hence, $\langle I, X\rangle = \operatorname{trace} X$ and $\langle J, X\rangle = \sum_{i=1}^n \sum_{j=1}^n X_{ij}$.

We can also define $\vartheta(G)$ by the dual SDP:

**Proposition 3.**

$$\begin{aligned} \vartheta(G) = \min_{t, Z \in \mathbb{S}^n} \quad & t \\ s.t. \quad & Z \preceq tI \\ & Z_{ij} = 1, \quad \forall (i,j) \ \textit{such that} \quad (i = j \quad \textit{or} \quad ij \in \bar{E}). \end{aligned} \tag{3}$$

*This SDP can also be rewritten as an eigenvalue problem:*

$$\vartheta(G) = \min_{Z \in \mathcal{Z}} \ \lambda_{\max}(Z),$$

*where* $\mathcal{Z} := \{Z \in \mathbb{S}^n | \quad Z_{ii} = 1, \ \forall i \in [n]; \quad Z_{ij} = 1, \ \forall \, ij \in \bar{E}\}$.

*Proof.* We start to show that the two SDPs are dual from each other. To this end, we first rewrite the max SDP as a saddle point (max-min) problem:

$$\vartheta(G) = \sup_{X \succeq 0} \langle J, X\rangle \ + \ \inf_{t \in \mathbb{R}} \ t \cdot (1 - \langle I, X\rangle) \ + \ \sum_{ij \in E} \ \inf_{u_{ij} \in \mathbb{R}} u_{ij} \cdot X_{ij}$$

The dual problem is obtained by switching the order of sup and inf. By weak duality,

$$\vartheta(G) \leq \vartheta'(G) := \inf_{t \in \mathbb{R}, u_{ij} \in \mathbb{R}} \ t \ + \ \sup_{X \succeq 0} \ \Big\langle X, \ J - t\, I + \sum_{ij \in E} u_{ij} \ E_{i,j} \Big\rangle,$$

where $E_{i,j} = \frac{1}{2}(\boldsymbol{e}_i \boldsymbol{e}_j^T + \boldsymbol{e}_j \boldsymbol{e}_i^T)$ is the matrix with $\frac{1}{2}$ on the $(i,j)$- and $(j,i)$-coordinates and 0's elsewhere, so it holds $X_{ij} = \langle X, E_{ij}\rangle$ for $i \neq j$. The above supremum is finite (and has value 0) if and only if $J - t\, I + \sum_{ij \in E} u_{ij} \ E_{i,j} \preceq 0$. Hence,

$$\begin{aligned} \vartheta'(G) \ = \inf_{t \in \mathbb{R}, u_{ij} \in \mathbb{R}} \quad & t \\ s.t. \quad & J + \sum_{ij \in E} u_{ij} \ E_{i,j} \preceq t\, I, \end{aligned}$$

Then, the dual SDP of the proposition is obtained by making the change of variable $Z = J + \sum_{ij \in E} u_{ij} \ E_{i,j}$, which is a symmetric matrix with arbitrary entries on coordinates $(i,j)$ where $ij \in E$, and with ones elsewhere, that is, $Z \in \mathcal{Z}$. The formulation as an eigenvalue problem follows from the SDP-representation of $\lambda_{\max}(\cdot)$.

It remains to show that strong duality holds, so $\vartheta(G) = \vartheta'(G)$. We are going to show that both SDPs are strictly feasible, which also implies that they are also bounded and attain their optimal values (so we can safely write "max" and "min" in the formulations of $\vartheta(G)$ instead of "sup" and "inf").

The matrix $X = \frac{1}{n} I_n$ is clearly strictly feasible for the primal SDP (the maximization problem). For the dual SDP, we observe that $Z \prec tI$ iff $\lambda_{\max}(Z) < t$. We can thus take an arbitrary matrix $Z \in \mathcal{Z}$, and choose $t > \lambda_{\max}(Z)$, so the pair $(Z, t)$ is strictly feasible for the dual SDP.

□

In the exercises, we will give yet another alternative SDP formulation of $\vartheta(G)$, which can be derived *in a systematic way*, as a relaxation from the integer quadratic programming formulation of $\alpha(G)$. We are now ready to prove the

**Theorem 4** (Lovász's Sandwich theorem)**.**

$$\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G).$$

*Proof.* For the first inequality, $\alpha(G) \leq \vartheta(G)$, let $S$ be a maximal stable set, and denote by $\boldsymbol{e}_S$ the incidence vector of $S$: $\boldsymbol{e}_S$ is the $\{0,1\}$-vector of size $n$ with a one on the $i$th coordinate iff $i \in S$. Then, we can see that $X = \frac{1}{|S|}\, \boldsymbol{e}_S \boldsymbol{e}_S^T$ is feasible for (2). Indeed, $X \succeq 0$ because it is defined as a tensor product of two vectors, $\operatorname{trace} X = \frac{1}{|S|} \operatorname{trace} \boldsymbol{e}_S \boldsymbol{e}_S^T = \frac{1}{|S|} \boldsymbol{e}_S^T \boldsymbol{e}_S = 1$, and for all $ij \in E$, we have ($i \notin S$ or $j \notin S$) because $S$ is a stable, so $(\boldsymbol{e}_S \boldsymbol{e}_S^T)_{ij} = 0 \implies X_{ij} = 0$. This shows that $\vartheta(G) \geq \langle J, X \rangle$, since the optimal value of the SDP is at least as large as the value of the particular solution $X$:

$$\vartheta(G) \geq \langle J, X \rangle = \frac{1}{|S|} \boldsymbol{e}_S^T J \boldsymbol{e}_S = \frac{1}{|S|} (\boldsymbol{e}_S^T \mathbf{1})^2 = \frac{1}{|S|} (|S|)^2 = |S| = \alpha(G).$$

We will proceed similarly for the second inequality, by identifying a feasible solution of value $\bar{\chi}(G)$ for the dual SDP. Let $C_1, \ldots, C_k$ be a minimal $k-$clique cover of $G$, and denote by $\boldsymbol{e}_{C_j}$ the incidence vector of $C_j$. We claim that $Z = k\,I - \frac{1}{k} \sum_{j=1}^k (k\boldsymbol{e}_{C_j} - \mathbf{1})(k\boldsymbol{e}_{C_j} - \mathbf{1})^T$ and $t = k$ are feasible for the SDP (3). Indeed, $t\,I - Z = \frac{1}{k} \sum_{j=1}^k (k\boldsymbol{e}_{C_j} - \mathbf{1})(k\boldsymbol{e}_{C_j} - \mathbf{1})^T$ is a sum of rank-one positive semidefinite matrices, and $t\,I \succeq Z$. Then, by observing that $\sum_{j=1}^k \boldsymbol{e}_{C_j} = \mathbf{1}$ (because the $C_j$'s form a partition of $V$), we can expand the expression of $Z$:

$$\begin{aligned} Z = k\,I - \frac{1}{k} \sum_{j=1}^k (k\boldsymbol{e}_{C_j} - \mathbf{1})(k\boldsymbol{e}_{C_j} - \mathbf{1})^T &= k\,I - k \sum_{j=1}^k \boldsymbol{e}_{C_j} \boldsymbol{e}_{C_j}^T + 2J - J \\ &= k\,(I - \sum_{j=1}^k \boldsymbol{e}_{C_j} \boldsymbol{e}_{C_j}^T) + J. \end{aligned}$$

Finally, the elements of the matrix $U := (I - \sum_{j=1}^k \boldsymbol{e}_{C_j} \boldsymbol{e}_{C_j}^T)$ are

$$U_{ij} = \begin{cases} -1 & \text{if } i \neq j \text{ belong to the same clique;} \\ 0 & \text{otherwise,} \end{cases}$$

so the diagonal elements of $Z$ are $Z_{ii} = J_{ii} = 1$, and for $ij \in \bar{E}$, the vertices $i$ and $j$ must belong to different cliques, so $Z_{ij} = J_{ij} = 1$. This shows $Z \in \mathcal{Z}$, and $\vartheta(G) \leq t = k = \bar{\chi}(G)$. □

We will see in the exercises that Semidefinite Programming can also be used to compute a maximum stable set in a perfect graph $G$. The algorithm needs to solve $n+1$ times the Lovász's SDP. Note that there is no known algorithm to compute a maximum stable set for perfect graphs without SDP. It is also possible to write an algorithm based on the Lovász's $\vartheta$-function to compute a minimum coloring of perfect graphs.

## 2 Maxcut SDP

Let $G$ be a simple graph with weights $w_{ij} \geq 0, \forall\, ij \in E$.

> **Definition 6** (cut)**.** A *cut* of $G$ is a partition of $V$ in two node sets $S$ and $\bar{S}$. The weight of a cut is the sum of the weights of the cut edges:
>
> $$\operatorname{cut}(S, \bar{S}) = \sum_{\substack{ij \in E \\ i \in S, j \notin S}} w_{ij}$$

The maximum cut problem asks to find the cut of maximum weight. From now on, we assume without loss of generality that $G$ is the complete graph, since adding edges of weight 0 does not change the maximum cut.

We represent a cut by a vector $\boldsymbol{x} \in \{-1, 1\}^n$, where $x_i = 1$ if $i \in S$, and $x_i = -1$ if $i \in \bar{S}$. Then, we have

$$1 - x_i x_j = \begin{cases} 2 & \text{if } \{i, j\} \text{ is a cut-edge} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\operatorname{cut}(S, \bar{S}) = \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - x_i x_j) = \frac{1}{4} \sum_{1 \leq i,j \leq n} w_{ij}(1 - x_i x_j)$ (there is an additional $\frac{1}{2}$-factor because each edge is counted twice in the sum). So the maximum cut problem can be formulated as

$$\begin{aligned} \underset{\boldsymbol{x}}{\textbf{maximize}} \quad & \frac{1}{4} \sum_{i,j} w_{ij}(1 - x_i x_j) \\ & \boldsymbol{x} \in \{-1, 1\}^n. \end{aligned}$$

To formulate an SDP relaxation, we introduce a matrix variable $X$, and we would like that $X_{ij} = x_i x_j$. To this end, we use the following lemma

**Lemma 5.** *The matrix $X \in \mathbb{S}^n$ satisfies $X_{ij} = x_i x_j$ for some vector $\boldsymbol{x} \in \{-1, 1\}^n$ if and only if*

$$X \succeq 0, \quad \textbf{diag}\,(X) = \mathbf{1}, \quad \textit{and} \quad \textbf{rank}\,(X) = 1.$$

*Proof.* We know that positive semidefinite matrices of rank 1 are of the form $X = \boldsymbol{u}\boldsymbol{u}^T$ for some vector $\boldsymbol{u} \in \mathbb{R}^n$, that is, $X_{ij} = u_i u_j$. Moreover, we have $(\boldsymbol{u}\boldsymbol{u}^T)_{ii} = (u_i)^2$, so $X_{ii} = 1 \iff u_i \in \{-1, 1\}$. □

If $X$ satisfies the property of this lemma, then it holds

$$\operatorname{cut}(S, \bar{S}) = \frac{1}{4} \sum_{i,j} w_{ij}(1 - x_i x_j) = \frac{1}{4} \langle W, J - X \rangle,$$

where $W$ is the symmetric matrix such that both $W_{ij}$ and $W_{ji}$ are set to the weight $w_{ij}$ of the edge $\{i, j\}$. If follows that the maximum cut problem is equivalent to the following optimization problem:

$$\begin{aligned} \underset{X \in \mathbb{S}^n}{\textbf{maximize}} \quad & \frac{1}{4} \langle W, J - X \rangle \\ & \textbf{diag}\,(X) = \mathbf{1} \\ & X \succeq 0 \\ & \textbf{rank}\,(X) = 1. \end{aligned}$$

The above problem *is not* an SDP, because of the nonconvex rank-one constraint. However, we otain an

SDP relaxation by removing that constraint:

$$\begin{aligned} \underset{X \in \mathbb{S}^n}{\textbf{maximize}} \quad & \frac{1}{4}\langle W, J - X\rangle \\ & \textbf{diag}\,(X) = \mathbf{1} \\ & X \succeq 0 \end{aligned} \tag{4}$$

Denote the optimal value of this relaxation by SDP. Since this is a relaxation (we removed a constraint, hence we optimize over a larger set of matrices), we have

$$\mathrm{maxcut}(G) \leq \mathrm{SDP}\,.$$

Note: In the exercises, we will give an alternative formulation for the MAXCUT SDP, which relies on the Laplacian matrix of the graph $G$, and can be used to derive analytic bounds on $\mathrm{maxcut}(G)$.

In a seminal paper, Goemans and Williamson showed that it is also possible to use the SDP (4) to derive an approximation algorithm for the maximum cut problem. We next present this randomized algorithm, which relies on projections on a random hyperplane:

1. Compute a solution $X^*$ of the SDP (4).
2. Compute a decomposition $X^* = H^T H$ (for example, a Cholesky decomposition). Denote the columns of $H$ by $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_n$. Note that the constraint $X^*_{ii} = 1$ implies $\boldsymbol{h}_i^T \boldsymbol{h}_i = 1$. Hence, the $\boldsymbol{h}_i$'s have unit norm.
3. Draw a vector $\boldsymbol{r}$ uniformly at random over the unit sphere of $\mathbb{R}^n$. To do this, one can draw independently some $z_i \sim \mathcal{N}(0,1)$ for $i = 1, \ldots, n$, and then take $\boldsymbol{r} = \frac{1}{\|\boldsymbol{z}\|}\boldsymbol{z}$.
4. Finally, return the cut defined by $S = \{i \in V : \boldsymbol{r}^T \boldsymbol{h}_i > 0\}$. (Or, equivalently, define the cut through the vector $x_i = \mathrm{sign}(\boldsymbol{r}^T \boldsymbol{h}_i)$, $\forall i \in [n]$.)

**Theorem 6** (Goemans & Williamson)**.** *Let $(S, \bar{S})$ be the (random) cut returned by the above random projection algorithm. Then,*

$$\mathbb{E}[\mathrm{cut}(S, \bar{S})] \geq \alpha \ \mathrm{SDP} \geq \alpha \ \mathrm{maxcut}(G),$$

*where $\alpha \simeq 0.87856$.*

The proof of this theorem is based on the following lemma:

**Lemma 7.** *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors on the unit sphere on $\mathbb{R}^n$, and let $\boldsymbol{r}$ be a random vector drawn uniformly at random on the sphere. Denote by $H$ be the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T \boldsymbol{r} = 0\}$. Then, the probability that $H$ separates $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to $\frac{\theta}{\pi}$, where $\theta = \arccos(\boldsymbol{u}^T \boldsymbol{v})$ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.*

*Proof.* (Sketch) We can reason in the two-dimensional subspace which contains $\boldsymbol{u}$ and $\boldsymbol{v}$, and for a suitable basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ of this subspace, it holds $\boldsymbol{u} = \boldsymbol{e}_1$ and $\boldsymbol{v} = \cos(\theta)\boldsymbol{e}_1 + \sin(\theta)\boldsymbol{e}_2$. Then, it is easy to see that the projection of $\boldsymbol{r}$ on this subspace is a vector of the form $\boldsymbol{r} = \rho\big(\cos(\alpha)\boldsymbol{e}_1 + \sin(\alpha)\boldsymbol{e}_2\big)$, where $\rho \leq 1$ and $\alpha$ is drawn unifomly at random in $[0, 2\pi]$. Finally, the hyperplane defined by $\boldsymbol{r}$ separates $\boldsymbol{u}$ and $\boldsymbol{v}$ iff $\boldsymbol{r}$ lies in a two-sided cone of angle $\theta$, which occurs with probability $\frac{2\theta}{2\pi}$. □

We are now ready to prove the theorem:

*Proof.* The expected weight of the cut $(S,\bar{S})$ is

$$\begin{aligned}\mathbb{E}[\mathrm{cut}(S,\bar{S})] &= \sum_{ij\in E} w_{ij}\mathbb{P}[\{i,j\}\text{ belongs to the cut set}]\\ &= \sum_{ij\in E} w_{ij}\frac{\arccos(\boldsymbol{h_i}^T\boldsymbol{h}_j)}{\pi}\end{aligned}$$

Now we multiply and divide the $(i,j)$th term of this sum by $\frac{1}{2}(1-\boldsymbol{h_i}^T\boldsymbol{h}_j)=\frac{1}{2}(1-X^*_{ij})$:

$$\mathbb{E}[\mathrm{cut}(S,\bar{S})] = \sum_{ij\in E}\frac{1}{2}w_{ij}(1-X^*_{ij})\frac{2\arccos(\boldsymbol{h_i}^T\boldsymbol{h}_j)}{\pi(1-\boldsymbol{h_i}^T\boldsymbol{h}_j)}$$

A straigtforward analysis shows that $\alpha := \inf_{\theta\in[0,\pi]}\frac{2}{\pi}\frac{\theta}{1-\cos(\theta)}\simeq 0.87856$. Hence,

$$\mathbb{E}[\mathrm{cut}(S,\bar{S})] \geq \alpha\sum_{ij\in E}\frac{1}{2}w_{ij}(1-X^*_{ij}) = \alpha\sum_{1\leq i,j\leq n}\frac{1}{4}w_{ij}(1-X^*_{ij}) = \alpha\,\frac{1}{4}\langle W, J-X^*\rangle = \alpha\ \mathrm{SDP}\,.$$

□

# 3 SDP relaxations for nonconvex QCQPs (with binary variables)

In the previous sections, we have derived SDP relaxations for the stable set problem and the maximum cut problem. In fact, both relaxations could have been obtained in a systemmatic manner, by using a *general recipe* that allows to obtain an SDP relaxation for any QCQP.

Let us consider an optimization problem of the form

$$\begin{aligned}\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}}\quad & \boldsymbol{x}^TQ_0\boldsymbol{x}+\boldsymbol{c}_0^T\boldsymbol{x}+q_0 \\ s.t.\quad & \boldsymbol{x}^TQ_i\boldsymbol{x}+\boldsymbol{c}_i^T\boldsymbol{x}+q_i\lesseqgtr 0,\quad \forall i\in[m],\end{aligned}\tag{5}$$

where the data $(Q_i,\boldsymbol{c}_i,q_i)_i$ is of appropriate size, and the symbol $\lesseqgtr$ replaces any of $\{\leq,=,\geq\}$. Note that this problem is not convex in general. In particular, it allows for binary constraints of the form $x_i\in\{0,1\}$, which can be formulated as equivalent quadratic equality constraints:

$$x_i\in\{0,1\} \iff x_i^2=x_i \iff \boldsymbol{x}^TQ\boldsymbol{x}+\boldsymbol{c}^T\boldsymbol{x}=0 \text{ for } Q=\boldsymbol{e}_i\boldsymbol{e}_i^T, \boldsymbol{c}=-\boldsymbol{e}_i.$$

We can use semidefinite programming to construct a relaxation of (5). To do this, one possibility is to introduce an auxilary variable $X=\boldsymbol{x}\boldsymbol{x}^T$: Prolem (5) is equivalent to

$$\begin{aligned}\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}}\quad & \langle Q_0,X\rangle+\boldsymbol{c}_0^T\boldsymbol{x}+q_0 \\ s.t.\quad & \langle Q_i,X\rangle+\boldsymbol{c}_i^T\boldsymbol{x}+q_i\lesseqgtr 0,\quad \forall i\in[m],\\ & X=\boldsymbol{x}\boldsymbol{x}^T.\end{aligned}\tag{6}$$

this problem is not convex because of the constraint $X=\boldsymbol{x}\boldsymbol{x}^T$. However, we obtain an SDP if we relax this constraint to $X\succeq\boldsymbol{x}\boldsymbol{x}^T$, which can be expressed as an LMI by using a Schur complement:

$$X\succeq\boldsymbol{x}\boldsymbol{x}^T \iff \begin{bmatrix} X & \boldsymbol{x}\\ \boldsymbol{x}^T & 1\end{bmatrix}\succeq 0.$$

Therefore, we obtain the following result:

**Proposition 8.** *The SDP*

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad & \langle Q_0, X\rangle + \boldsymbol{c}_0^T\boldsymbol{x} + q_0 \\ s.t. \quad & \langle Q_i, X\rangle + \boldsymbol{c}_i^T\boldsymbol{x} + q_i \leqslant 0, \quad \forall i \in [m], \\ & \begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \succeq 0 \end{aligned} \tag{7}$$

*is a relaxation of Problem* (5)*. Its optimal value gives a lower bound for the original nonconvex QCQP.*

We observe that binary variables $x_i \in \{0,1\} \iff x_i^2 = x_i$ result in constraints of the form

$$X_{ii} = x_i$$

in the SDP. Similarly, a binary variable $x_j \in \{-1,1\} \iff x_j^2 = 1$ yields the constraint $X_{jj} = 1$.

There is an alternative way to interprete this SDP, by proceeding as we did for MAXCUT. Indeed, note that we obtain an exact reformulation of Problem (5) when we add the nonconvex constraint $\textbf{rank} \begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} = 1$ to the SDP. To see this,

$$\begin{aligned} \begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \succeq 0 \text{ is of rank } 1 \quad & \iff \quad \begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u} \\ \alpha \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \alpha \end{bmatrix}^T \text{ for some } \boldsymbol{u} \in \mathbb{R}^n, \alpha \in \mathbb{R} \\ & \iff \quad X = \boldsymbol{u}\boldsymbol{u}^T \text{ for } \boldsymbol{u} = \alpha\boldsymbol{x}, \quad \alpha = \pm 1 \\ & \iff \quad X = \boldsymbol{x}\boldsymbol{x}^T. \end{aligned}$$

## 4 Completely positive formulation for binary QPs

It was shown by Burer that we can even obtain an exact conic reformulation for QPs with binary variables. To do this, we need to introduce the cone of *copositive matrices*:

**Definition 7** (Copositive cone)**.** The cone of $n \times n$ copositive matrices is

$$\mathcal{C}_n := \{X \in \mathbb{S}^n \mid \boldsymbol{u}^T X \boldsymbol{u} \geq 0,\ \forall \boldsymbol{u} \in \mathbb{R}^n_+\}.$$

Note that the copositive cone only differs from the semidefinite cone from the restriction $\boldsymbol{u} \geq \boldsymbol{0}$. We also introduce the cone of *completely positive matrices*:

**Definition 8** (Completely positive cone)**.** The cone of $n \times n$ completely positive matrices is

$$\mathcal{C}_n^* := \{\sum_{k=1}^{q} \boldsymbol{u}_k\boldsymbol{u}_k^T \mid q \in \mathbb{N},\ \boldsymbol{u}_k \in \mathbb{R}^n_+,\ \forall k \in [q]\}.$$

The next proposition gives important properties above these 2 cones:

**Proposition 9** (Properties of $\mathcal{C}_n$ and $\mathcal{C}_n^*$)**.**

*(i) We require at most $q \leq n(n+1)/2$ vectors to decompose a completely positive matrix:*

$$X \in \mathcal{C}_n^* \iff \exists \ell \leq \frac{1}{2}n(n+1), \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_\ell \in \mathbb{R}_+^n : X = \sum_{j=1}^{\ell} \boldsymbol{x}_i \boldsymbol{x}_i^T.$$

*(ii) The following inclusions hold:*

$$\mathcal{C}_n^* \subseteq \left(\mathbb{S}_+^n \cap \mathbb{R}_+^{n\times n}\right) \subseteq \mathbb{S}_+^n \subseteq \left(\mathbb{S}_+^n + \mathbb{R}_+^{n\times n}\right) \subseteq \mathcal{C}_n.$$

*(iii) The cones $\mathcal{C}_n$ and $\mathcal{C}_n^*$ are proper, and dual from each other.*

*Proof.* (i). We have $\mathcal{C}_n^* = \mathbf{cone}\,\{\boldsymbol{u}\boldsymbol{u}^T | \boldsymbol{u} \in \mathbb{R}_n^+\}$. The affine dimension of $\{\boldsymbol{u}\boldsymbol{u}^T | \boldsymbol{u} \in \mathbb{R}_n^+\} \subseteq \mathbb{S}^n$ is less than $n(n+1)/2$ (the affine dimension of $\mathbb{S}^n$), so Caratheodory's theorem tells us that every element of $\mathcal{C}_n^*$ can be expressed as a conic combination of $q \leq n(n+1)/2$ elements of $\{\boldsymbol{u}\boldsymbol{u}^T | \boldsymbol{u} \in \mathbb{R}_n^+\}$.

(ii). The two inclusions in the middle are trivial, so we only prove the first and the last inclusions. Let $X \in \mathcal{C}_n^*$. Then $X$ is positive semidefinite, because it can be written as a sum of rank-one positive semidefinite matrices. Moreover, the elements of $X = \sum_k \boldsymbol{u}_k \boldsymbol{u}_k^T$, where $\forall k\, \boldsymbol{u}_k \geq \mathbf{0}$, are clearly nonnegative. This shows: $\mathcal{C}_n^* \subseteq \mathbb{S}_+^n \cap \mathbb{R}_+^{n\times n}$. Now, let $X \in \left(\mathbb{S}_+^n + \mathbb{R}_+^{n\times n}\right)$, that is, $X = Y + Z$ for some matrices $Y \succeq 0$ and $Z \geq 0$. Then, for all $\boldsymbol{u} \geq \mathbf{0}$, it holds $\boldsymbol{u}^T X \boldsymbol{u} = \boldsymbol{u}^T Y \boldsymbol{u} + \boldsymbol{u}^T Z \boldsymbol{u}$, and $\boldsymbol{u}^T Y \boldsymbol{u} \geq 0$ because $Y \succeq 0$, $\boldsymbol{u}^T Z \boldsymbol{u} \geq 0$ because it is a sum of products of nonnegative numbers. This shows $\left(\mathbb{S}_+^n + \mathbb{R}_+^{n\times n}\right) \subseteq \mathcal{C}_n$.

(iii). The cone $\mathcal{C}_n$ is clearly closed, and convex (it is the intersection of infinitely many halfspaces). Its interior is nonempty, which can be seen from $\mathbb{S}_+^n \subseteq \mathcal{C}_n \implies \mathbf{int}\,\mathbb{S}_+^n = \mathbb{S}_{++}^n \subseteq \mathbf{int}\,\mathcal{C}_n$. To see that the cone is pointed, assume that $X \in \mathcal{C}_n$ and $-X \in \mathcal{C}_n$, that is, $\boldsymbol{u}^T X \boldsymbol{u} = 0, \forall \boldsymbol{u} \geq \mathbf{0}$. We can choose $\boldsymbol{u} = \boldsymbol{e}_i$, which gives $\boldsymbol{u}^T X \boldsymbol{u} = X_{ii} = 0$, so the diagonal elements of $X$ are 0. Then, choosing $\boldsymbol{u} = \boldsymbol{e}_i + \boldsymbol{e}_j$, we get $\boldsymbol{u}^T X \boldsymbol{u} = X_{ii} + X_{jj} + 2X_{ij} = 0 \implies X_{ij} = 0$, so the off-diagonal elements of $X$ must be 0, too.

This shows that $\mathcal{C}_n$ is proper. Now, we show that $\mathcal{C}_n$ is the dual cone of $\mathcal{C}_n^*$, which also implies that $\mathcal{C}_n^*$ is the dual of $\mathcal{C}_n$, and that $\mathcal{C}_n^*$ is a proper cone, because we know that the dual cone of a proper cone is proper.

$$\begin{aligned}
Y \in \mathrm{dual}(\mathcal{C}_n^*) &\iff \forall X \in \mathcal{C}_n^*, \langle X, Y\rangle \geq 0 \\
&\iff \forall q \in \mathbb{N}, \forall \boldsymbol{u}_1, \ldots, \boldsymbol{u}_q \in \mathbb{R}_+^n,\ \sum_{k=1}^{q} \boldsymbol{u}_k^T Y \boldsymbol{u}_k \geq 0 \\
&\iff \forall \boldsymbol{u} \geq \mathbf{0},\ \boldsymbol{u}^T Y \boldsymbol{u} \geq 0 \\
&\iff Y \in \mathcal{C}_n
\end{aligned}$$

□

Now, consider a mixed-integer QP of the form

$$\begin{aligned}
\underset{\boldsymbol{x}\in\mathbb{R}^n}{\mathbf{minimize}} \quad & \boldsymbol{x}^T Q \boldsymbol{x} + \boldsymbol{c}^T \boldsymbol{x} \\
s.t. \quad & \boldsymbol{a}_i^T \boldsymbol{x} = b_i,\ \forall i \in [m] \\
& \boldsymbol{x} \geq \mathbf{0} \\
& x_i \in \{0,1\},\ \forall i \in B,
\end{aligned} \tag{8}$$

where $B$ is a subset of $[n]$. By applying the general recipe of Proposition 8, this problem admits the following

SDP relaxation:

$$\begin{aligned}
\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad & \langle Q, X\rangle + \boldsymbol{c}^T\boldsymbol{x} \\
s.t. \quad & \boldsymbol{a}_i^T\boldsymbol{x} = b_i,\ \forall i\in[m] \\
& \boldsymbol{x}\geq \boldsymbol{0} \\
& X_{ii} = x_i,\ \forall i \in B, \\
& \begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \succeq 0
\end{aligned} \tag{9}$$

Burer proposed to make two modifications to this SDP. First, he adds the (redundant) quadratic equalities $(\boldsymbol{a}_i^T\boldsymbol{x})^2 = b_i^2$ in the original problem formulation, which yields the new constraints $\boldsymbol{a}_i^T X \boldsymbol{a}_i = b_i^2$ in the SDP. Then, he observes that thanks to the constraint $\boldsymbol{x}\geq\boldsymbol{0}$, the matrix $\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}$ is not only positive semidefinite, but also completely positive. With these two modifications, the relaxation becomes exact !

**Theorem 10** (Burer)**.** *Under some mild assumption (which can always be achieved without loss of generality), the completely positive program*

$$\begin{aligned}
\underset{\boldsymbol{x}\in\mathbb{R}^n}{\textbf{minimize}} \quad & \langle Q, X\rangle + \boldsymbol{c}^T\boldsymbol{x} \\
s.t. \quad & \boldsymbol{a}_i^T\boldsymbol{x} = b_i,\ \forall i\in[m] \\
& \boldsymbol{a}_i^T X \boldsymbol{a}_i = b_i^2,\ \forall i\in[m] \\
& X_{ii} = x_i,\ \forall i \in B, \\
& \begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \succeq_{\mathcal{C}_n^*} 0
\end{aligned} \tag{10}$$

*is* equivalent *to the mixed-integer QP* (8)*.*

We are not going to prove this result, but we will prove a special case of this result for a QP formulation of the maximum stable set problem in the exercises.

Of course, we can not expect to solve the above completely positive program in polynomial time, as (8) contains many NP-hard optimization problems as special cases. Hence, this shows that completely positive programming is intractable. This is a nice example to show that convex optimization problems can be NP-hard. The difficulty comes from the fact that it is $NP$-hard to *separate* the completely positive cone, that is, to decide whether a matrix $X$ is completely positive, or to return a separating hyperplane.

Nevertheless, the completely positive formulation can be used to construct hierarchies of SDPs that converge to the optimal value of (8). The first level of this hierarchy is the well known *doubly nonnegative relaxation*, in which the constraint $\begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \succeq_{\mathcal{C}_n^*} 0$ is replaced by

$$X \in \mathbb{S}_+^n \cap \mathbb{R}_+^{n\times n} \iff \begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} X & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \geq 0.$$

The doubly nonnegative relaxation is known to be exact for $n\leq 4$, but is only an approximation for $n\geq 5$.