Greedy Algorithm for Independence Systems

Input: independence system $(E, \mathcal{F})$ given by oracle, $c \in \mathbb{R}^E$
Output: $F \in \mathcal{F}$

1. sort $E = \{e_1, e_2, \ldots, e_n\}$ such that $c(e_1) \geq c(e_2) \geq \cdots \geq c(e_n)$;
2. set $F := \emptyset$;
3. for $i = 1$ to n do: if $F \cup \{e_i\} \in \mathcal{F}$, then set $F := F \cup \{e_i\}$;

Theorem 16.11.

Let $(E, \mathcal{F})$ be an independence system and $c \in \mathbb{R}^E$. Let $G(E, \mathcal{F}, c)$ be the cost of the solution found by the Greedy Algorithm. Then,

$$\underbrace{q(E, \mathcal{F})}_{=1 \text{ for matroids}} \leq \frac{G(E, \mathcal{F}, c)}{\mathrm{OPT}(E, \mathcal{F}, c)} \leq 1 \text{ .} \quad \text{trivial}$$

There is a cost function $c \in \mathbb{R}^E$ for which the lower bound is attained.

Proof:. . . $\square$

Corollary: Greedy is optimal $\forall c$ on matroids

Proof:

Let $G_n \subseteq E$ be the greedy solution

Let $O_n \subseteq E$ be an optimal solution

Let $E_j := \{e_1, e_2, \ldots, e_j\}$ $\quad G_j := G_n \cap E_j$ $\quad G_0, O_0 = \emptyset$

$O_j := O_n \cap E_j$

Then $c(G_n) = \sum_{j=1}^{n} (|G_j| - |G_{j-1}|) \, c(e_j)$

$c(e_j)$ only counted when e_j is taken

(*)
$$= |G_1| c(e_1) - |G_0| c(e_0) + |G|_2 c(e_2) - |G_1| c(e_2) + |G_3| c(e_3) - |G_2| c(e_3) + \ldots + |G_n| c(e_n) - |G_{n-1}| c(e_n)$$

$$= \sum_{j=1}^{n} |G_j| \underbrace{(c(e_j) - c(e_{j+1}))}_{=: d_j \geq 0} = \sum_{j=1}^{n} |G_j| d_j$$

G_j is basis of $E_j \Rightarrow |G_j| \geq \rho(E_j)$

$$\geq \sum_{j=1}^{n} \rho(E_j)\, d_j$$

$$\frac{\rho(E_j)}{r(E_j)} \geq q(E,\mathcal{F}) \Rightarrow \rho(E_j) \geq q(E,\mathcal{F}) \cdot r(E_j)$$

$$\geq q(E,\mathcal{F}) \sum_{j=1}^{n} r(E_j) \cdot d_j$$

$O_j \in \mathcal{F}, \; O_j \subseteq E_j \Rightarrow r(E_j) \geq |O_j|$

$$\geq q(E,\mathcal{F}) \sum_{j=1}^{n} |O_j| \cdot d_j$$

$$= q(E,\mathcal{F}) \sum_{j=1}^{n} \left(|O_j| - |O_{j-1}|\right) c(e_j)$$

as in (*)

$$= q(E,\mathcal{F}) \cdot c(O_n)$$

Show that the lower bound is tight when $(E,\mathcal{F})$ is not a matroid

$(E,\mathcal{F})$ no matroid $\Rightarrow \exists\, F \in \mathcal{F}$ with bases B_1 and B_2

$$\frac{|B_1|}{|B_2|} = q(E,\mathcal{F})$$

let $c(e) : \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$

Sort E s.t. $c(e_1) \geq c(e_2) \geq \dots \geq c(e_n)$

and that $B_1 = \{e_1, \dots, e_{|B_1|}\}$

$\Rightarrow G(E,\mathcal{F},c) = |B_1|$

$\mathrm{OPT}(E,\mathcal{F},c) = |B_2|$

Corollary : $\boxed{(E,\mathcal{F}) \text{ is a matroid} \iff \text{Greedy computes an optimal solution for all } c}$

Polyhedral Description of Matroids

Theorem 16.12.

Let $(E, \mathcal{F})$ be a matroid with rank function $r : 2^E \to \mathbb{Z}_{\geq 0}$. The convex hull of the incidence vectors of all elements of $\mathcal{F}$ is equal to

$$\left\{ x \in \mathbb{R}^E \,\middle|\, x \geq 0, \sum_{e \in X} x_e \leq r(X) \text{ for all } X \subseteq E \right\} . \quad =: P$$

Proof:... □

conv. hull $(\mathcal{F})$ = convex hull of all χ^F, $F \in \mathcal{F}$

every such χ^F fulfills the inequalities of P

$\Rightarrow$ conv. hull $(\mathcal{F}) \subseteq P$

"$\supseteq$" show that all vertices of P are incidence vectors χ^F of some $F \in \mathcal{F}$

let $c \in \mathbb{R}^E$ w.l.o.g. $c \geq 0$ (negative elements are not taken)

$$c(G_n) = \sum_{j=1}^{n} r(E_j) \cdot d_j \geq \sum_{j=1}^{n} \left(\sum_{e \in E_j} x_e \right) d_j$$

greedy solution

see proof of Thm 16.11
use that $(E, \mathcal{F})$ is a matroid

inequality from LP

$$= \sum_{j=1}^{n} \left(\sum_{e \in E_j} x_e - \sum_{e \in E_{j-1}} x_e \right) c(e_j)$$

as in (*) in the proof of Thm 16.11

$$= \sum_{j=1}^{n} x_{e_j} c(e_j) = \sum_{e\in E} c(e)\cdot x_e = c^T x$$

$\Rightarrow c(G_n) \geq OPT(LP) \overset{\text{LP is a relaxation}}{\Rightarrow} c(G_n) = OPT(LP)$

$\Rightarrow$ vertices of P are incidence vectors of greedy solutions

$\Rightarrow$ conv. hull $(\mathcal{F}) \supseteq P$ $\square$

Conclusion: We can solve LPs of the form

$\max c^T x$, $\sum_{e\in X} x_e \geq r(X) \;\forall X\subseteq E$, $x \geq 0$

with the greedy algorithm if r is the rank function of a matroid

Weighted Matroid Intersection Problem

Given: Two matroids $(E, \mathcal{F}_1)$, $(E, \mathcal{F}_2)$ on a common finite set E, $c \in \mathbb{R}^E$.

Task: Find $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ maximizing $c(X)$.

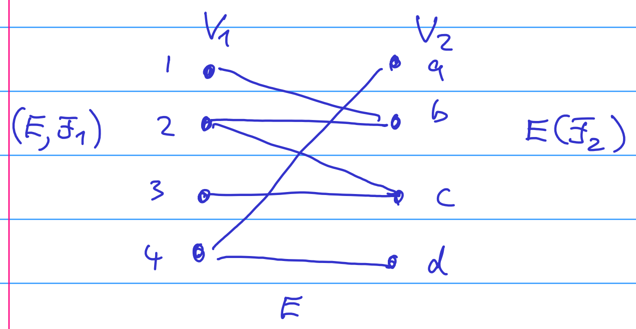
Examples

- Maximum weight matching problem on bipartite graphs.
- Given a digraph D with weights on the arcs, find a branching (forest in which every node has indegree at most one) of maximum weight.

In the following, we restrict to the unit-weight case, i. e., the problem of finding $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ maximizing $|X|$.

We refer to this problem as the Matroid Intersection Problem.

Matching problem on bipartite graphs



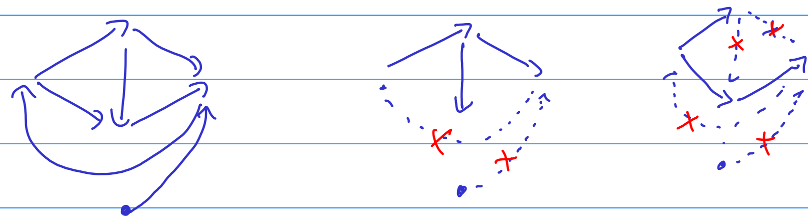
$F \subseteq E$, $F \in \mathcal{F}_i$ $:\Longleftrightarrow$ every vertex in V_i is incident to at most one edge from F

Bases of $(E, \mathcal{F}_1)$ are $\{ 1b \quad 2b \quad 3c \quad 4a \}$ can be exchanged
$\qquad\qquad\qquad\qquad 2c \qquad\quad 4d$

$F \in \mathcal{F}_1 \cap \mathcal{F}_2 \Longleftrightarrow F$ is a matching

Branchings in digraph

Example:



$(E, \mathcal{F}_1)$: $F \in \mathcal{F}_1 \Leftrightarrow$ underlying undirected edges from F are a forest

$(E, \mathcal{F}_2)$: $F \in \mathcal{F}_2 \Leftrightarrow$ every vertex has at most one ingoing arc

as for bipartite matching

Matroid Intersection: Preliminaries

Notation: Let $(E, \mathcal{F})$ be a matroid. For $X \in \mathcal{F}$ and $f \in E$ let

$$C(X, f) := \begin{cases} \emptyset & \text{if } X \cup \{f\} \in \mathcal{F}, \\ \text{unique circuit in } X \cup \{f\} & \text{otherwise.} \end{cases}$$

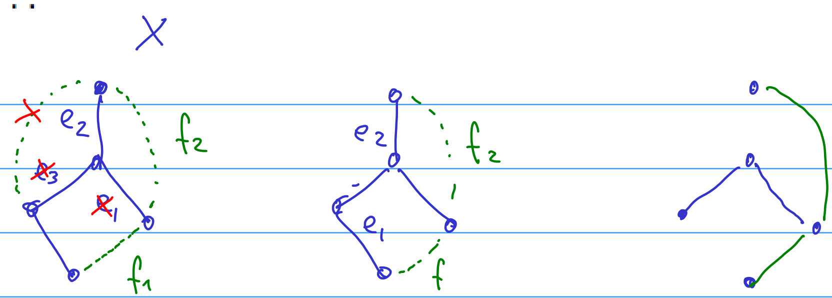
Lemma 16.13.

Let $(E, \mathcal{F})$ be a matroid, $X \in \mathcal{F}$, $e_1, \ldots, e_s \in X$, and $f_1, \ldots, f_s \notin X$ with

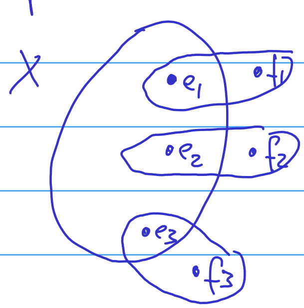
1. $e_k \in C(X, f_k)$ for $k = 1, \ldots, s$ and
2. $e_j \notin C(X, f_k)$ for $j < k$.

Then, $(X \setminus \{e_1, \ldots, e_s\}) \cup \{f_1, \ldots, f_s\} \in \mathcal{F}$.

Proof:. . . $\square$



Proof:



wo $e_j \in C(X, f_k)$ $\quad j < k$

$\Rightarrow (X \setminus \{e_1, \ldots, e_s\}) \cup \{f_1, \ldots, f_s\} \in \mathcal{F}$

Let $X_r := (X \setminus \{e_1, \ldots, e_r\}) \cup \{f_1, \ldots, f_r\}$ $\quad 0 \le r \le s$

Use induction on r

$r = 0$: $\quad X_0 = X \in \mathcal{F}$ ✓

$r-1 \to r$:

by inductive assumption $X_{r-1} \in \mathcal{F}$

if $X_{r-1} \cup \{f_r\} \in \mathcal{F} \Rightarrow X_r \in \mathcal{F}$

otherwise $C(X_{r-1}, f_r)$ is the unique circuit in $X_{r-1} \cup \{f_r\}$

and $C(X, f_r) \underset{\uparrow \atop (ii)}{\subseteq} X \setminus \{e_1, \dots, e_{r-1}\} \cup \{f_r\} \subseteq X_{r-1} \cup \{f_r\}$

circuits are unique $\Rightarrow C(X_{r-1}, f_r) = C(X, f_r)$

$\overset{(i)}{\Rightarrow} e_r \in C(X_{r-1}, f_r)$

$\Rightarrow X_r = (X_{r-1} \setminus \{e_r\}) \cup \{f_r\} \in \mathcal{F} \quad \square$

$\uparrow$

circuits are minimally dependent