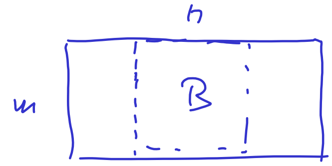
# Unimodularity and Integrality




B basis

**Definition 14.16.**

A matrix $A$ of full row rank is unimodular if $A$ is integral and each basis of $A$ has determinant $-1$ or $1$.

**Theorem 14.17 (Veinott/Dantzig 1968).**

Let $A \in \mathbb{Z}^{m\times n}$ be a matrix of full row rank. Then $\underbrace{\{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}}_{P}$ is integral for every integral vector $b \in \mathbb{R}^m$ $\Longleftrightarrow$ $A$ is unimodular.

this makes it a property of A only

Proof: . . .

"$\Leftarrow$" Let $A$ be unimodular

Let $b \in \mathbb{Z}^m$ and $\bar{x}$ a vertex of $P$

$\hookrightarrow$ exists ($P$ is pointed)

$\overset{\text{Def vertex}}{\Longrightarrow}$ $\exists$ $n$ lin. indep. constraints that are tight at $\bar{x}$

$\Longrightarrow$ columns $A_j$ of $A$ with $\bar{x}_j > 0$ are lin. indep.

extend them to basis $B$ of $A$

$\overset{\text{ass.}}{\Longrightarrow}$ $\det B \in \{-1, +1\}$

$\Longrightarrow$ $\bar{x}_B = B^{-1} b$ is integral and contains all $\bar{x}_j > 0$

$\uparrow$

La 14.15

"$\Rightarrow$" Let $B$ be a basis of $A$. Show $\det(B) \in \{-1, 1\}$

Let $v \in \mathbb{Z}^m$. Show that $\boxed{B^{-1}v \text{ is integral}}$ $\nearrow$ La 14.15

Let $y$ be integral with $y + B^{-1}v \geq 0$ $\quad y \in \mathbb{Z}^m$ integral

Let $b := B(y + B^{-1}v) = \underbrace{By}_{\text{integral}} + \underbrace{v}_{\text{integral}}$

Extend $y + B^{-1}v$ to vector $z \in \mathbb{R}^n$ by adding 0-components

$\Rightarrow Az = B(y + B^{-1}v) = b, \quad z \geq 0$

$\Rightarrow z \in P$ and $n$ lin indep constraints are tight in $z$

the $m$ equations and the $n-m$ $z_i = 0$

$\Rightarrow z$ is a vertex of $P$

$\Rightarrow$ (Assumption) $z$ is integral $\Rightarrow B^{-1}v$ integral

$y$ integral

$y + B^{-1}v = z_B$ $\square$

## Total Unimodularity

### Definition 14.18.

Matrix $A$ is totally unimodular (TUM) if all of its square submatrices have determinant $-1, 0$ or $1$.

Thus, a TUM-matrix has all entries in $\{-1, 0, 1\}$.

### Observation 14.19.

adding slack variables does not change TUM

$A \in \mathbb{R}^{m \times n}$ is TUM $\iff$ $[A\ I_m]$ is unimodular.

Proof: Exercise. $\square$

### Theorem 14.20 (Hoffman/Kruskal 1956).

Let $A \in \mathbb{Z}^{m \times n}$. Then the polyhedron $\overbrace{\{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq 0\}}^{P}$ is integral for every integral vector $b \in \mathbb{R}^m$ $\iff$ $A$ is totally unimodular.

Proof: ... $\square$

$P$ integral $\iff \bar{P} := \{ x \in \mathbb{R}^n \mid [A\ I_m] \cdot z = b \quad x \geq 0$

add slack variables

is integral

$\overset{14.17}{\iff} [A\ I_m]$ is unimodular

$\overset{14.19}{\Rightarrow}$ statement $\square$

## Polyhedra without Non-Negativity Constraints

Note: For a given $b$ it may be true that $\{x \mid Ax \le b,\ x \ge 0\}$ is integral even if $A$ is not TUM ($\longrightarrow$ Exercise.)

**Theorem 14.21.**

Let $A \in \mathbb{R}^{m \times n}$ be TUM and $b \in \mathbb{R}^m$ be integral. Then the polyhedron defined by $Ax \le b$ is integral.

Proof: . . . $\square$

This Theorem is often applied in order to prove the integrality of polyhedra. But how can we detect whether a given matrix is TUM?

In 1980, Seymour proved a deep Theorem which gives a polynomial-time test. In particulare, he showed that the backbone of every TUM-matrix are so-called network matrices, defined below.

Proof Thm 14.21

Show that every minimal face contains an integral vector

So let $F$ be an $\subseteq$-minimal face of $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$

$\Rightarrow$ $F = \{x \mid A'x = b'\}$ for a subsystem of $Ax \le b$

Obs. 14.6 and $\operatorname{rank}(A') = \operatorname{rank}(A)$

Reorder the columns of $A'$ such that $A' = [B\ N]$ $B$ basis of $A'$

$\underset{\substack{\uparrow \\ A \text{ TUM}}}{\Rightarrow}$ $\bar{x} := \begin{pmatrix} B^{-1}b' \\ 0 \end{pmatrix}$ is integral and $\bar{x} \in F$ $\square$

## Node-Arc Incidence-Matrices

Let us start with an example of network matrices:

**Theorem 14.22 (Poincaré, 1900).**

Let $A$ be a matrix with entries in $\{-1, 0, 1\}$, where each column has at most one $+1$ and at most one $-1$. Then $A$ is TUM.

Proof: … □

**Corollary 14.23.**

Let $D = (V, E)$ be a directed graph and let $A$ be the node-arc incidence matrix of $D$ with entries

$$a_{ve} := \begin{cases} +1 & \text{if } v \text{ is the head of } e, \\ -1 & \text{if } v \text{ is the tail of } e, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A$ is TUM.

This covers flow problems

Example: …

Proof Thm. 14.22

induction on the size $k$ of $k \times k$ submatrices $N$ of $A$

$k = 1$ ✓

inductive step to $k \geq 2$

case 1: $N$ contains a column with only $0$

$\Rightarrow \det(N) = 0$

case 2: $N$ contains a column with only one entry $\neq 0$ $a_{ij}$

Laplace expansion of $\det(N)$ along this column

$\Rightarrow |\det(N)| = |a_{ij}| \cdot |\det(N')|$

$N'$: submatrix after deleting row $i$ and column $j$

$\overset{\text{ind. assumption}}{\Rightarrow} |\det(N')| \in \{0, -1, 1\}$

$\Rightarrow \det(N) \in \{0, 1, -1\}$

case 3: all columns of $N$ have $\geq 2$ entries $\neq 0$

$\Rightarrow$ (ass. on $A$) all columns have exactly one $+1$ and exactly one $-1$

$\Rightarrow$ sum of rows of $N$ $= \mathcal{O}$

$\Rightarrow$ $\det(N) = 0$ $\square$

Corollary: Assumptions of Thm 14.22 can be generalized to

(1) every column has at most 2 entries $\neq 0$

(2) the rows can be partitioned into 2 sets $R_1, R_2$ (one may be empty) such that

a)
$$\left.\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}\right\} \Rightarrow \text{one row in } R_1\text{, the other in } R_2$$
signs are the same

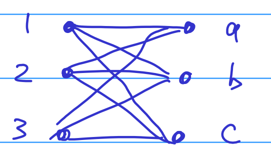
b)
$$\left.\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right\} \Rightarrow \text{both rows lie in the same set}$$
signs are opposite

we need only change the proof of case 3 by taking $+$ all rows in $R_1$ $-$ all rows in $R_2$

this gives again $\mathcal{O}$

Example for this generalization:

vertex-edge incidence matrices of bipartite graphs are TUM




| | | 1a | 1b | 1c | 2a | 2b | 2c | 3a | 3b | 3c |
|---|---|---|---|---|---|---|---|---|---|---|
| $R_1$ | 1 | 1 | 1 | 1 | | | | | | |
| | 2 | | | | 1 | 1 | 1 | | | |
| | 3 | | | | | | | 1 | 1 | 1 |
| $R_2$ | a | 1 | | | 1 | | | 1 | | |
| | b | | 1 | | | 1 | | | 1 | |
| | c | | | 1 | | | 1 | | | 1 |

# Network Matrices

**Definition 14.24.**

Let $D = (V, E)$ be a digraph and $T = (V, E')$ be a spanning tree o~~f $D$~~ f.
↳ need not be arcs of $D$
Then $M \in \{-1, 0, 1\}^{E' \times E}$ with entries

$$M_{e',(u,v)} := \begin{cases} +1 & \text{if the } (u,v)\text{-path in } T \text{ uses } e' \text{ in forward direction,} \\ -1 & \text{if the } (u,v)\text{-path in } T \text{ uses } e' \text{ in backward direction,} \\ 0 & \text{otherwise.} \end{cases}$$

is called network matrix (w.r.t. $D$ and $T$).

Example: ...

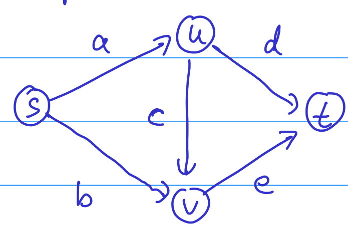
**Theorem 14.25 (Tutte, 1965).**

Network Matrices are TUM.

Proof: ... □

Example:

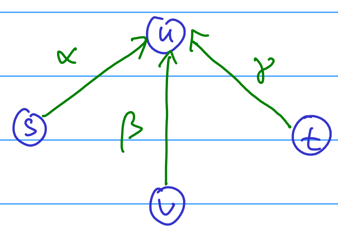
$D$




$$A = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline s & -1 & -1 & & & \\ u & 1 & & -1 & -1 & \\ v & & 1 & 1 & & -1 \\ t & & & & 1 & 1 \end{array}$$

TUM

$T$




$$M = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline \alpha & 1 & 1 & & & \\ \beta & & -1 & -1 & & 1 \\ \gamma & & & & -1 & -1 \end{array}$$

$L$ = node-arc incidence matrix of $T$

$$L = \begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline s & -1 & & \\ u & 1 & 1 & 1 \\ v & & -1 & \\ t & & & -1 \end{array}$$

TUM

Es gilt $\boxed{L \cdot M = A}$ Excercise

Proof Thm 14.25

Let $M$ be a network matrix for digraph $D$ and tree $T$

Let $A$ be the node-arc incidence matrix of $D$

Let $L$ . . . . . . . . . $T$

Choose a leaf from $T$ and delete the row in $A$ and $L$

$\longrightarrow \bar{A}$ and $\bar{L}$

Thm 14.22 $\Rightarrow [\bar{L} \ \bar{A}]$ TUM

ADM I : a set of columns of a node arc incidence matrix (with one row deleted) is a basis

$\Leftrightarrow$ the columns form a spanning tree

network simplex

here we need only "$\Leftarrow$" (this can be shown by preorder traversal of the tree (Left, Right, Root) and ordering the vertices and arcs in the order of visiting / traversing them

So $\bar{L}$ is a basis of $[\bar{L} \bar{A}]$

Claim $\boxed{\bar{L}^{-1} \bar{A} \text{ is TUM}}$

$$\bar{L}^{-1} \cdot \underbrace{[\bar{L} \ \bar{A}]}_{\text{TUM}} = [I \ \bar{L}^{-1} \bar{A}]$$

$\Updownarrow$

Excercise

multiplying a TUM matrix with a unimodular matrix gives a TUM matrix

$\bar{L}^{-1} \bar{A}$ is TUM

Now we use $\boxed{LM = A}$ $\Rightarrow$ $\tilde{L} M = \tilde{A}$

$\Rightarrow$ $M = \tilde{L}^{-1} \tilde{A}$

$\Rightarrow$ $M$ is TOM $\square$