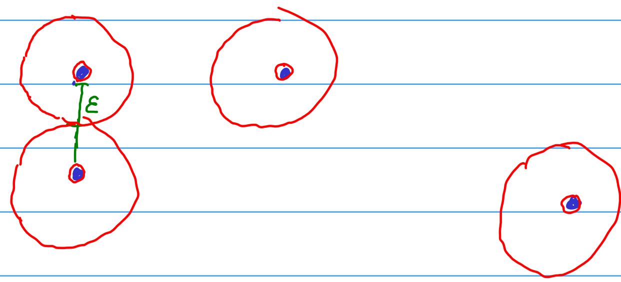
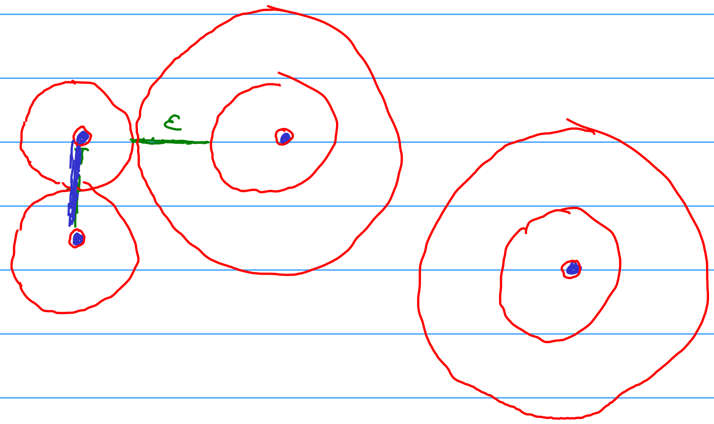
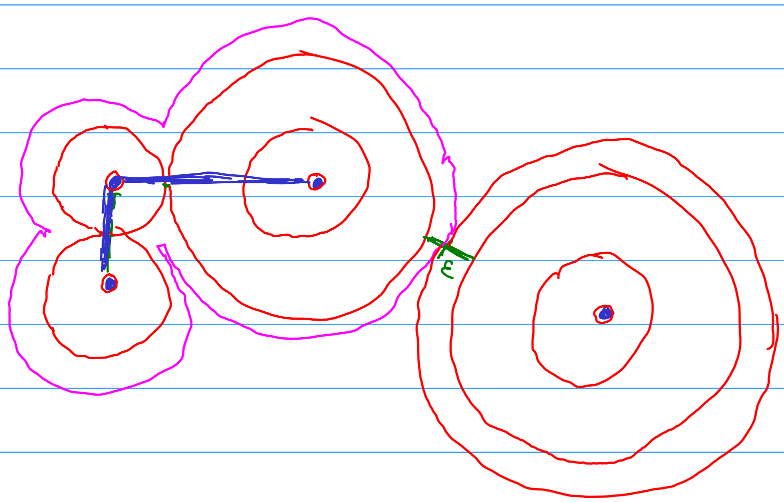
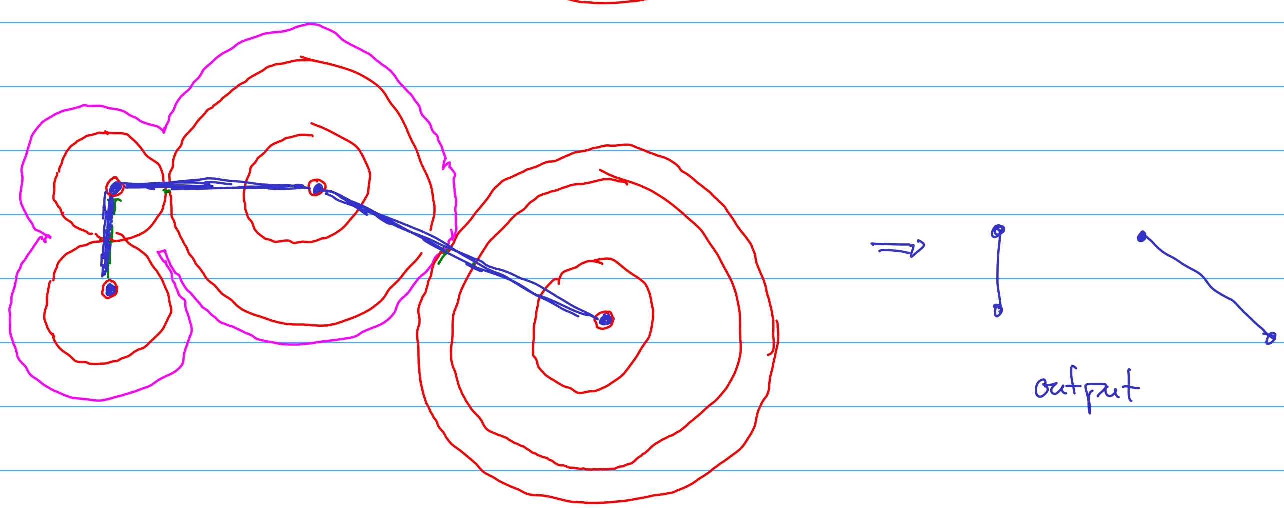
Goemans - Williamson Algorithm

Example









Proof Theorem 13.32

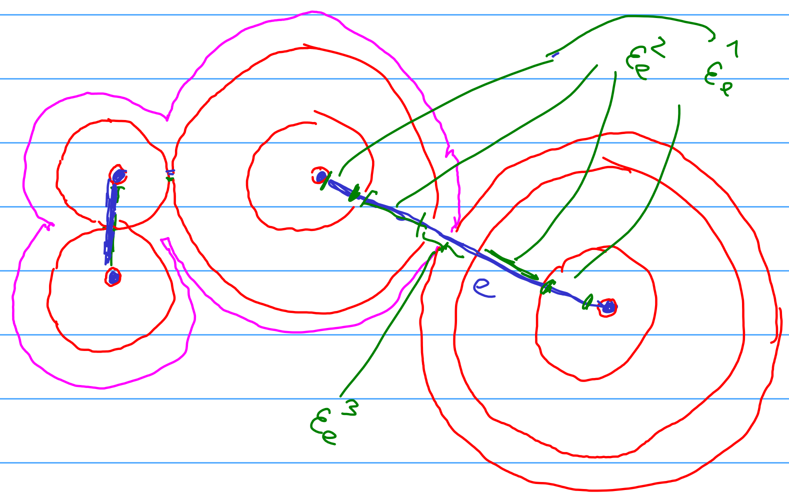
By construction, the algorithm terminates with an even forest $\mathcal{F}^*$ since $|V|$ is even

Let $\mathcal{T}^k$ = node sets of trees
ε^k = minimum ε
} in iteration k, $k = 1, \dots, t$

Consider edge $e \in \mathcal{F}^*$

Then $c_e = c_e^1 + c_e^2 + \dots + c_e^t$

↑ growth over all derivations of dual variables involving e



e enters $\mathcal{F}^*$ in some iteration, say s

let $e = vw$ $\quad v \in T_i \in \mathcal{T}^s$, $\quad w \in T_j \in \mathcal{T}^s$

$\Rightarrow c_e^s = \varepsilon^s(\text{parity}(T_i) + \text{parity}(T_j))$

and $c_e^k = 0$ for $k > s$ (then $T_i = T_j$)

in earlier iterations $k < s$

$\Rightarrow c_e^k$ grows also by $\varepsilon^k(\text{parity}(T_i) + \text{parity}(T_j))$

$$\text{so} \quad c_e^k = \begin{cases} \varepsilon^k(\text{parity}(T_i) + \text{parity}(T_j)) & \text{if } T_i \neq T_j \\ 0 & T_i = T_j \end{cases}$$

Claim 1: $\sum_{e \in F^*} c_e^k \leq 2 \cdot \varepsilon^k \underbrace{\left| \{ T \in \mathcal{T}^k \mid |T| \text{ odd} \} \right|}_{}$ $\forall k$

$=: LB^k$

= lower bound on growth of all dual variables in iteration k

Claim 1 $\Rightarrow$ $\sum_{e \in F^*} c_e \leq 2 \cdot LB^k \leq 2$ total growth of dual variables in all iterations

$= 2 \cdot \left(\sum_{v \in V} y_v^* + \sum_{D \text{ ungerade}} Y_D^* \right)$

$\parallel$

$$\boxed{c(F^*) \leq 2 \cdot \text{value of dual solution } (y^*, Y^*)}$$

$\Rightarrow$ proof of Theorem

Proof of Claim 1

Consider iteration k

$F^* \longrightarrow \bar{F}^*$

shrink all $T \in \mathcal{T}^k$ to one node v_T

Label nodes in $\bar{F}^*$ odd / even if T was odd / even

let V^{odd} = set of odd nodes in $\bar{F}^*$

V^{even} = ... even ... ...

Claim 2: $\sum_{v \text{ odd}} \deg(v) \leq 2 |V^{odd}|$ in $\bar{F}^*$

Proof Claim 2 (may delete isolated vertices)

$\sum_{v \text{ odd}} \deg(v) = \underbrace{\sum_{v} \deg(v)}_{= 2|E| \leq 2|V| \text{ since } \bar{F}^* \text{ is a forest}} - \sum_{v \text{ even}} \deg(v)$

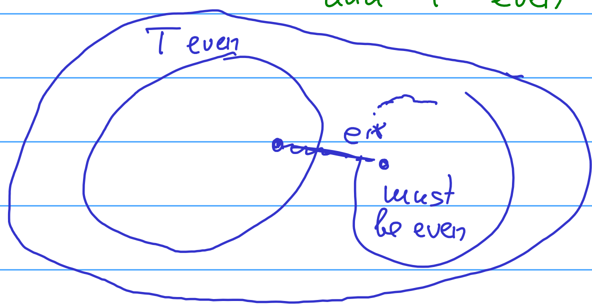
$$\leq 2|V^{odd}| + 2|V^{even}| - 2|V^{even}|$$

every node in V^{even} has degree ≥ 2

v_T even in $\mathcal{F}^*$ with degree 1 $\Rightarrow$ v_T leaf in $\mathcal{T}^*$

$\Rightarrow$ T has only one outgoing edge $e^* \in \mathcal{F}^*$

and T even



$\mathcal{F}^*$ even forest

$\Rightarrow$ e^* is even

contradiction to construction of $\mathcal{F}^*$

So $\sum_{v \text{ odd}} \deg(v) \leq 2|V^{odd}|$ $\Rightarrow$ Claim 2

Now $$\sum_{e \in \mathcal{F}^*} c_e^k = \varepsilon^k \sum_{T_i - e - T_j} (\text{parity}(T_i) + \text{parity}(T_j))$$

edges between different trees T_i, T_j

count 1 for every edge leaving odd T_i

$$= \varepsilon^k \sum_{v \text{ odd in } \mathcal{F}^*} \deg(v)$$

$$\overset{\text{Claim 2}}{\leq} 2 \cdot \varepsilon^k |V^{odd}|$$

$$= 2 \cdot \varepsilon^k |\{T \in \mathcal{T}^k \mid |T| \text{ odd}\}| \quad \square$$

So we have a 2-approximation algorithm for perfect min weight matchings in complete graphs in which c_e fulfill the triangle inequality

Postman Problem

Given: Connected graph $G = (V, E)$ with edge weights $c \in \mathbb{R}^E_{\geq 0}$.

Task: Find a postman tour, i. e., a closed path traversing every edge in G at least once. Minimize the total weight of the path.

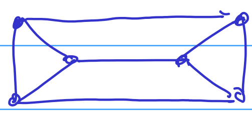
Remarks:

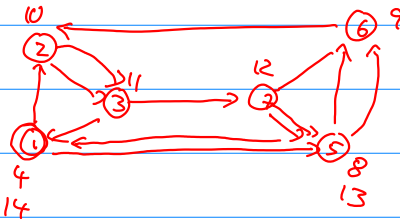
- Equivalently, instead of minimizing the total weight, one can minimize the cost of the "extra" edge-traversals.
- If there is a postman tour with no extra edge traversals, then that tour is optimal.

Definition 13.33.
A closed edge-simple path P such that $E(P) = E$ is called an Euler tour.

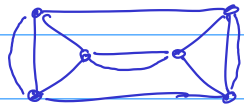
Theorem 13.34.
A connected graph G has an Euler tour if and only if every node of G has even degree. □

Example





better solution



Integer Programming Formulation

Let $x_e \in \mathbb{Z}_{\geq 0}$ be the number of extra traversals of edge e for each $e \in E$.

By the last theorem, we can formulate the postman problem as follows:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & x(\delta(v)) \equiv |\delta(v)| \mod 2 \qquad \text{for all } v \in V \\ & x \in \mathbb{Z}^E_{\geq 0} \end{aligned}$$

Remarks:

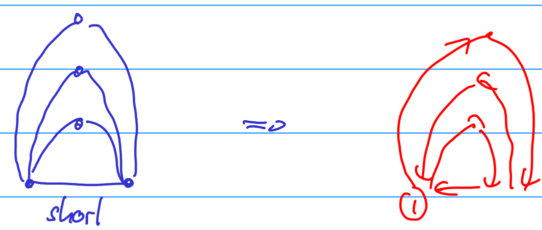
- There is an optimal solution with $x_e \in \{0, 1\}$ for all $e \in E$.
- Thus, the task is to find $J \subseteq E$ such that

$$|\delta(v) \cap J| \equiv |\delta(v)| \mod 2 \qquad \text{for all } v \in V.$$

  We call such a J a postman set.

can fulfil side constraint with $x_e \leq 1$

Example



T-Joins

Definition 13.35.

Let $G = (V, E)$ be a graph, and let $T \subseteq V$ such that $|T|$ is even. A *T-join* of G is a set J of edges such that

$$|J \cap \delta(v)| \equiv |T \cap \{v\}| \mod 2 \qquad \text{for all } v \in V.$$

That is, the odd-degree nodes of (V, J) are exactly the elements of T.

set of vertices of odd degree in postman problem

Optimal T-join problem

Given: Graph $G = (V, E)$ with weights $c \in \mathbb{R}^E$ and $T \subseteq V$ with $|T|$ even.

Task: Find a T-join J of G such that $c(J)$ is minimum.

generalizes postman problem

Examples:

- Postman sets. Let $T := \{v \in V \mid |\delta(v)| \text{ odd}\}$. Then the T-joins are precisely the postman sets.
- s-t-paths. Let $T := \{s, t\} \subseteq V$. Then, every T-join contains the edge set of an s-t-path.

Solving the Optimal T-Join Problem

Lemma 13.36.

Let J' be a T'-join of G. Then J is a T-join of G if and only if $J \triangle J'$ is a $T \triangle T'$-join of G.

Proof:... □

Lemma 13.37.

Every minimal (w.r.t. inclusion) T-join is the union of the edge-sets of $|T|/2$ edge-disjoint simple paths, which join the nodes in T in pairs.

Proof:... □

Lemma 13.38.

Suppose that $c \geq 0$. Then there is an optimal T-join that is the union of $|T|/2$ edge-disjoint shortest paths joining the nodes of T in pairs.

Proof:... □

Proof of Lemma 13.36

"$\Rightarrow$" Let J' be a T'-join of G and J be a T-join of G

Let $v \in V$. Then

$|(J \triangle J') \cap \delta(v)|$ even

$\Leftrightarrow |J \cap \delta(v)| \equiv |J' \cap \delta(v)| \bmod 2$

call v red if $v \in J \cap \delta(v)$

green if $v \in J' \cap \delta(v)$

then $\underbrace{\#(\text{green} \triangle \text{red})}_{\text{even}} = \#\text{red} - \#\text{redgreen} + \#\text{green} - \#\text{redgreen}$

$\Leftrightarrow$ same parity

$\Leftrightarrow \#\text{red}, \#\text{green}$ same parity

$\Leftrightarrow$ v in both T and T' or in none of them

$\Leftrightarrow \quad v \notin T \triangle T'$

$\Rightarrow \quad J \triangle J'$ is $T \triangle T'$-join

"$\Leftarrow$" replace J by $J \triangle J'$ and T by $T \triangle T'$
and apply the same argument
Note that $(J \triangle J') \triangle J' = J$

$(T \triangle T') \triangle T' = T$ $\square$