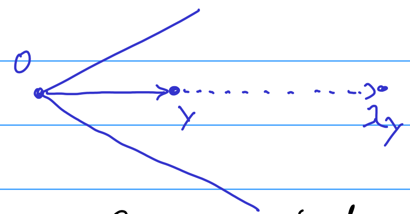
Exercise groups on Wednesday

Exercises on homepage: coga.math.tu-berlin.de/adm2

Def **Cone** $C \subseteq \mathbb{R}^n$ falls

$y \in C \Rightarrow \lambda y \in C \quad \forall \lambda \geq 0$

$C \Rightarrow 0 \in C$



cone C is pointed if has an extreme point

0 is the only possible extreme point

L $y \neq 0$, $y \in C \Rightarrow \underbrace{\tfrac{1}{2} y}_{y_1} \in C$, $\underbrace{\tfrac{3}{2} y}_{y_2} \in C$

$\Rightarrow y = \tfrac{1}{2} y_1 + \tfrac{1}{2} \cdot y_2$

Recession Cones

Definition 9.6.

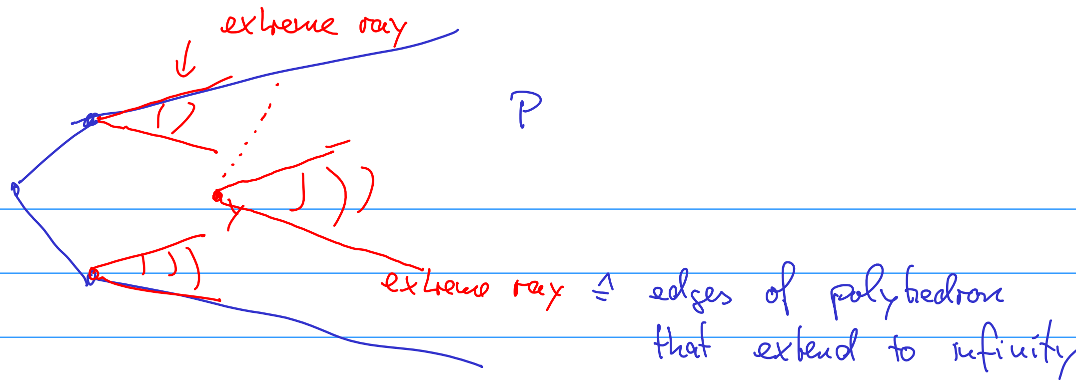
Consider a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ and $y \in P$. The recession cone of P (at y) is the set

$$\{d \in \mathbb{R}^n \mid y + \lambda \cdot d \in P \text{ for all } \lambda \geq 0\} \ .$$

The non-zero elements of the recession cone are the rays of P.

Remarks:

- Notice that
$$\begin{aligned} &\{d \in \mathbb{R}^n \mid y + \lambda \cdot d \in P \text{ for all } \lambda \geq 0\} \\ &\quad = \{d \in \mathbb{R}^n \mid A \cdot (y + \lambda \cdot d) \geq b \text{ for all } \lambda \geq 0\} \\ &\quad = \{d \in \mathbb{R}^n \mid A \cdot d \geq 0\} \ . \end{aligned}$$
- The definition of the recession cone of P is independent of $y \in P$.
- The recession cone of P is a polyhedral cone.
- The recession cone of $P = \{x \in \mathbb{R}^n \mid A \cdot x = b, \, x \geq 0\}$ is
$$\{d \in \mathbb{R}^n \mid A \cdot d = 0, \, d \geq 0\} \ .$$



Extreme Rays

Observation.
A non-empty polyhedron P has a vertex if and only if its recession cone is pointed. In this case we also say that P is pointed.

Definition 9.7.

- **a** A non-zero element x of a polyhedral cone $C \subseteq \mathbb{R}^n$ is an extreme ray if there are $n-1$ linearly independent constraints that are active at x.
- **b** An extreme ray of the recession cone of a polyhedron P is also called an extreme ray of P.

Remark: Up to multiplication with positive factors, there are only finitely many extreme rays of a polyhedron.

Characterization of Unbounded LPs

Theorem 9.8.
Let $C := \{x \in \mathbb{R}^n \mid {a_i}^T \cdot x \geq 0,\ i = 1, \ldots, m\}$ a pointed polyhedral cone and $c \in \mathbb{R}^n$. The minimal cost $c^T \cdot x$ subject to $x \in C$ is equal to $-\infty$ if and only if there is an extreme ray d of C with $c^T \cdot d < 0$.

Proof: . . . □

The result also holds if C is a polyhedron with at least one extreme point.

Theorem 9.9.
Let $P \subseteq \mathbb{R}^n$ be a polyhedron with at least one extreme point and $c \in \mathbb{R}^n$. The minimal cost $c^T \cdot x$ subject to $x \in P$ is equal to $-\infty$ if and only if there is an extreme ray d of P with $c^T \cdot d < 0$.

Proof: . . . □

Remark: If the simplex method observes that an LP is unbounded, the corresponding jth basic direction is an extreme ray d with $c^T \cdot d < 0$.

Proof Thm. 9.8

"$\Leftarrow$" $\exists$ extreme ray d with $c^T d < 0$

$\Rightarrow$ $c^T x$ is unbounded from below

"$\Rightarrow$" let $c^T x$ be unbounded from below in C

$\Rightarrow$ $\exists\ x \in C$ mit $c^T x < 0$

$\exists\ x \in C$ mit $c^T x = -1$

Set $P = \{ x \in \mathbb{R}^n \mid \underbrace{a_i^T x \geq 0 \quad i = 1, \dots, m}_{\Leftrightarrow\ x \in C},\ c^T x = -1 \} \neq \emptyset$

C pointed $\overset{\text{Thm 9.5}}{\Rightarrow}$ $a_1, \dots, a_m$ span $\mathbb{R}^n$

$\Rightarrow$ $a_1, \dots, a_m$ lin. indep. in P

$\overset{\text{ADM 1}}{\Rightarrow}$ P hat extreme point d

$\Rightarrow$ $n-1$ lin ind. constraints from $a_i^T x \geq 0$ are active at d

$d \neq 0$ $\Rightarrow$ d is extreme ray of C $\square$

Proof Thm. 9.9.

"$\Leftarrow$" clear

"$\Rightarrow$" Consider $\min c^T x$ on P

$\min = -\infty$ $\Rightarrow$ dual is infeasible

$$\left.\begin{aligned} &\max\ p^T b \\ &\text{s.t. } p^T A = c^T \\ &\qquad p \geq 0 \end{aligned}\right\} \text{ no feasible solution}$$

replace objective by $p^T \cdot 0$

$\Rightarrow$ problem stays infeasible

Consider the corresponding primal

$$\min\ c^T x \quad \text{s.t. } \underbrace{Ax \geq 0}_{\text{recession cone } C \text{ of } P}$$

(see remark to Def. 9.6)

0 is feasible solution of this primal
dual infeasible } $\Rightarrow$ primal unbounded

Assumption: $Ax \geq b$ has extreme point

$\Rightarrow$ $\exists$ n lin indep. constraints

$\Rightarrow$ rows of A span $\mathbb{R}^n$

$\overset{\text{Thm 9.5}}{\Rightarrow}$ C is pointed

$\overset{\text{Thm 9.8}}{\Rightarrow}$ C contains extreme ray d with $c^T d < 0$

$c^T x$ unbounded on C

$\Rightarrow$ P contains ... $\square$

Resolution Theorem

Theorem 9.10.

Let $P := \{x \in \mathbb{R}^n \mid A \cdot x \geq b\} \neq \emptyset$ pointed. Let $x^1, \ldots, x^k$ be the extreme points and $w^1, \ldots, w^r$ a complete set of extreme rays of P. Then,

$$P = \underbrace{\left\{ \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j \mid \lambda_i, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}}_{=:Q} .$$

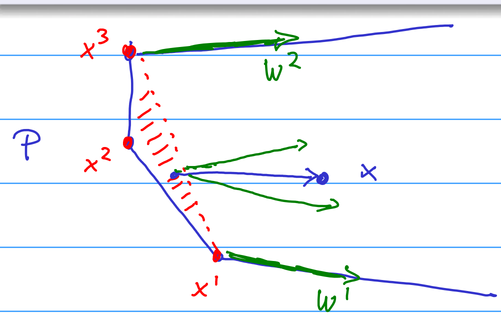
Proof: ... $\square$

Corollary 9.11.

A non-empty polytope is equal to the convex hull of its extreme points. $\square$

Corollary 9.12.

Every element of a pointed polyhedral cone is a non-negative linear combination of its extreme rays. $\square$



Proof Thm. 9.10

"$\supseteq$" $P \supseteq Q$ clear by convexity and definition of rays

"$\subseteq$" $P \subseteq Q$

assume by contradiction that there is $z \in P \setminus Q$

consider LP

variables

$$\max \sum_{i=1}^{k} 0 \lambda_i + \sum_{j=1}^{r} 0 \cdot \theta_j$$

$$\text{s.t.} \quad \sum_i \lambda_i x^i + \sum_j \theta_j w_j = z$$

$$\sum_i \lambda_i = 1$$

$$\lambda_i, \theta_j \geq 0$$

infeasible since $z \notin Q$

Take the dual

$$\min \ p^T z + q$$

$$\text{s.t.} \quad p^T x^i + q \geq 0 \quad \forall i$$

$$p^T w^j \geq 0 \quad \forall j$$

$$p, q \text{ unrestricted}, \quad p \in \mathbb{R}^n, \quad q \in \mathbb{R}^1$$

$p = 0$, $q = 0$ is feasible in the dual

$\Rightarrow$ dual LP is unbounded

$\Rightarrow$ $\exists$ solution (p, q) with $p^T z + q < 0$

$\Rightarrow$ $\underline{p^T z} < -q \leq \underline{p^T x^i} \quad \forall i$ (1)

first ineq. of dual

$p^T w^j \geq 0$ (2)

second ineq. of dual

For this fixed vector p, consider the LP

$$\min \ p^T x$$

$$\text{s.t.} \quad Ax \geq b$$

$z \in P$ $\Rightarrow$ z feasible solution of this LP

$= \{x \in \mathbb{R}^n \mid Ax \geq b\}$

Case 1 : LP hat finite optimal cost

Ass: P has extreme points $\xRightarrow{}$ optimal solution is attained at an extreme point, say x^i
ADM 1

But $p^T z \overset{(1)}{<} p^T x^i$, contradiction

Case 2 : LP is unbounded

$\overset{\text{Thm } 9.9}{\Longrightarrow}$ $\exists$ extreme ray w^j with $p^T w^j < 0$

$\Rightarrow$ contradiction to (2) $\square$

Converse to the Resolution Theorem

Definition 9.13.

A set $Q \subseteq \mathbb{R}^n$ is finitely generated if there are $x^1, \ldots, x^k, w^1, \ldots, w^r \in \mathbb{R}^n$ such that

$$Q = \left\{ \sum_{i=1}^{k} \lambda_i \cdot x^i + \sum_{j=1}^{r} \theta_j \cdot w^j \mid \lambda_i, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\} .$$

Remark: The Resolution Theorem states that a polyhedron with at least one extreme point is finitely generated (also true for general polyhedra).

Theorem 9.14.

A finitely generated set is a polyhedron. In particular, the convex hull of finitely many vectors is a polytope.

Proof: … $\square$

↳ Take $z \in \mathbb{R}^n$ and the LP

$$\max \sum_{i=1}^{k} 0 \cdot \lambda_i + \sum_{j=1}^{r} 0 \cdot \theta_j$$

$$\text{s.t.} \quad \sum_i \lambda_i x^i + \sum \theta_j w_j = z$$

$$\sum_i \lambda_i = 1$$

$$\lambda_i , \theta_j \geqslant 0$$

$z \in Q \iff$ LP is feasible and has finite optimal value

consider the dual LP

$$\begin{aligned} \min\ & p^T z + q \\ \text{s.t.}\ & p^T x^i + q \geq 0 \quad \forall i \\ & p^T w^j \geq 0 \quad \forall j \end{aligned}$$

$z \in Q \iff$ dual LP has finite optimal solution

convert the dual to standard form

write $p = p^+ - p^-$ $\quad q = q^+ - q^-$

introduce slack variables α_i, β_j

$$\Rightarrow \begin{aligned} \min\ & (p^+ - p^-)^T z + (q^+ - q^-) \\ \text{s.t.}\ & (p^+ - p^-)^T x^i + (q^+ - q^-) - \alpha_i = 0 \quad \forall i \\ & (p^+ - p^-)^T w^j - \beta_j = 0 \quad \forall j \\ & p^+, p^-, q^+, q^-, \alpha_i, \beta_j \geq 0 \end{aligned}$$

LPs in standard form have pointed feasibility domain

Our LP has finite optimum

$\underset{\text{Thm 9.9}}{\iff}$ no extreme ray $d = (p^+, p^-, q^+, q^-, \alpha_i, \beta_j)$ with $(p^+ - p^-)^T z + (q^+ - q^-) < 0$

$\iff$ $\underbrace{(p^+ - p^-)}_{\text{fixed}} z + \underbrace{(q^+ - q^-)}_{\text{fixed}} \geq 0 \qquad (*)$

for all extreme rays

$\iff$ $(*)$ holds if we restrict to a complete set of extreme rays

finitely many

Conclusion: $z \in Q \iff z$ fulfills the finitely many inequalities $(*)$

$\Rightarrow Q$ is a polyhedron $\square$

Representation of Polyhedra

Conclusion: There are two ways of representing a polyhedron:

i in terms of a finite set of linear constraints (outer representation);

ii as a finitely generated set, in terms of its extreme points and rays (inner representation).

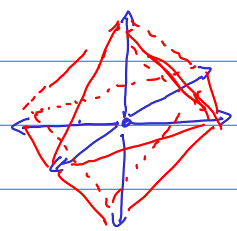
Remarks:

- Passing from one type of description to the other is, in general, a complicated computational task.
- One description can be small while the other one is huge. Examples:
  - An n-dimensional cube is given by $2n$ linear constraints and has 2^n extreme points.
  - A representation of the convex hull of the $2n$ points

$$e_1, -e_1, e_2, -e_2, \ldots, e_n, -e_n$$

in terms of linear constraints needs at least 2^n linear inequalities.

in $\mathbb{R}^3$



octahedron