Introduction to Linear and Combinatorial Optimization (ADM I)

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based on the 2011/12 course by Martin Skutella

TU Berlin

WS 2013/14

Dates for Tutorien:
Wednesday: 8–10, ... 16–18
(2 out of those)

General Remarks

- new flavor of ADM II — continuation of "new ADM I" from last semester
- lectures (Rolf Möhring):
  Thursday, 14:15 – 15:45, MA 043
  Friday, 14:15 – 15:45, MA 041
- exercise session (Torsten Gellert):
  Wednesday, 14:15 – 15:45, MA 041 (starting next week)
- tutorial sessions (Antje Lehmann), details see web page
- homework: set of problems every week (solve in groups of at most 3), sometimes programming exercises (details t.b.a.)
- "Scheinkriterium": 50 % of points from problem sets 1–6 and 7–12
- final oral exam (Modulabschlussprüfung) in summer during the semester break (details t.b.a.), written exam if 50 or more students take the course !

Outline

ADM I

ADM II

LP Theorie

Graphenalgo

Ganzzahlige Lin Opt

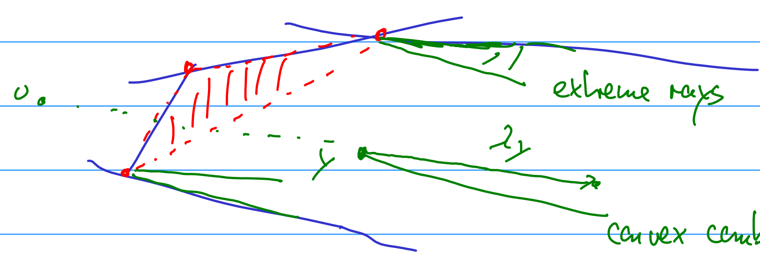
Verbindung zu algebraischen Konzepten

schnelle Näherungslösungen für NP-vollst. Probleme

Chapter 9:
Linear Programming & Polyhedral Theory

(cp. Bertsimas & Tsitsiklis, Chapters 2.8, 4.6–4.9)

First Goal: Representation Theorem for general polyhedra



every $y \in P$ has a representation as convex comb. of extreme points + non-neg comb of extreme rays

Fourier-Motzkin Elimination

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. If follows from Gaussian elimination that

$$\exists x \in \mathbb{R}^n : A \cdot x = b \quad \Longleftrightarrow \quad \nexists y \in \mathbb{R}^m : y^T \cdot A = 0^T,\ y^T \cdot b = -1$$

Aim: Derive analogous characterization for existence of x with $A \cdot x \leq b$.

Idea: Rewrite $A \cdot x \leq b$ equivalently (multiply rows by positive scalars) as

$$\begin{aligned} x_1 + (a_i')^T \cdot x' &\leq b_i && \text{for all } i = 1, \dots, m' \\ -x_1 + (a_i')^T \cdot x' &\leq b_i && \text{for all } i = m'+1, \dots, m'' \\ (a_i')^T \cdot x' &\leq b_i && \text{for all } i = m''+1, \dots, m \end{aligned}$$

with $x' = (x_2, \dots, x_n)^T$ and $(a_i')^T = i$th row of A with first entry deleted.

This system has a solution (x_1, x') if and only if there is $x' \in \mathbb{R}^{n-1}$ with

$$\begin{aligned} (a_j')^T \cdot x' - b_j \leq b_i - (a_i')^T \cdot x' && \text{for all } i = 1, \dots, m',\ j = m'+1, \dots, m'' \\ (a_i')^T \cdot x' \leq b_i && \text{for all } i = m''+1, \dots, m \end{aligned}$$

Example:

$$\left.\begin{aligned} x_1 - \tfrac{1}{4} x_2 &\leq -1 \\ x_1 + 2x_2 - x_3 &\leq -5 \end{aligned}\right\} \text{"+"} \qquad \text{coeff of } x_1 \text{ is positive}$$

$$\left.\begin{aligned} -x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &\leq -1 \\ -x_1 - \tfrac{2}{3}x_2 &\leq -1 \end{aligned}\right\} \text{"−"} \qquad \text{" negative}$$

$$\left.\begin{aligned} -x_2 - x_3 &\leq -1 \end{aligned}\right\} \text{"0"} \qquad \text{-"- } 0$$

all combinations of adding a "−" inequality to a "+" ineq.

$$\left.\begin{aligned} -\tfrac{3}{4}x_2 - \tfrac{1}{2}x_3 &\leq -2 \\ \tfrac{3}{2}x_2 + \tfrac{1}{2}x_3 &\leq -6 \\ -\tfrac{11}{12}x_2 &\leq -2 \\ \tfrac{4}{3}x_2 - x_3 &\leq -6 \end{aligned}\right\} \text{combinations}$$

$$\left.\begin{aligned} -x_2 - x_3 &\leq -1 \end{aligned}\right\} \text{"0" ineq.}$$

new system $\hat{=}$ $\boxed{uA \leq ub \qquad u \geq 0}$

models elementary row operations

Problem: # of inequalities may grow exponentially when this process is repeated

Farkas' Lemma

The system $A \cdot x \leq b$ has a solution x if and only if there is $x' \in \mathbb{R}^{n-1}$ with

$$(a'_i + a'_j)^T \cdot x' \leq b_i + b_j \quad \text{for all } i = 1, \ldots, m', j = m'+1, \ldots, m''$$
$$(a'_i)^T \cdot x' \leq b_i \quad \text{for all } i = m''+1, \ldots, m$$

The following theorem is known as Farkas' Lemma:

Theorem 9.1.
The system $A \cdot x \leq b$ has a solution x, if and only if there is no vector y satisfying $y \geq 0$, $y^T \cdot A = 0$ and $y^T \cdot b < 0$.

Proof: ... □

Corollary 9.2. standard form used by Simplex algo
The system $A \cdot x = b$ has a non-negative solution $x \geq 0$, if and only if there is no vector y satisfying $y^T \cdot A \geq 0$ and $y^T \cdot b < 0$. □

Remark: Farkas' Lemma also follows immediately from LP duality.

Proof: Thm 9.1

"$\Rightarrow$" Let $x \in \mathbb{R}^n$ with $Ax \leq b$
suppose there is $y \geq 0$ with $y^TA = 0$, $y^Tb < 0$
Then $0 > y^Tb \geq y^TAx = (y^TA) \cdot x = 0$

"$\Leftarrow$" Suppose that $Ax \leq b$ has no solution
Show by induction on # columns of A
that $\exists\, y \geq 0$ with $y^TA = 0$ and $y^Tb < 0$

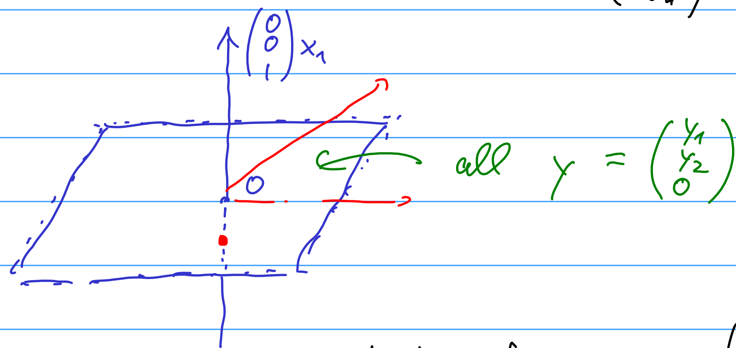
n = 1

$$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} x_1 \leq \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad \text{no solution}$$

$$y^T \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = 0 \quad \hat{=} \quad \text{subspace of } \mathbb{R}^m \text{ orthogonal to } \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

$x_1 \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \hat{=}$ line in $\mathbb{R}^m$ through 0

$\Rightarrow$ (coordinate transformation) $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$



no solution $\Rightarrow$ $b \notin$ subspace $x_1 \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$

and $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \ngeq 0$

└ otherwise $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \leq b$

$\Rightarrow \exists\, i_0$ with $b_{i_0} < 0$ $\quad i_0 \in \{1, \dots, m-1\}$

$\Rightarrow$ set $y_{i_0} = 1$, $y_i = 0$ otherwise $\Rightarrow y^T b < 0$

$y \geq 0$

inductive step $n-1 \to n$

$Ax \leq b$ has no solution

$\Rightarrow$ system $A'x' \leq b'$ with $n-1$ columns has no solution

↑

Fourier Motzkin

$\Rightarrow$ $\exists\, y' \geq 0$ with $y'^T A' = 0 \quad y'^T b' < 0$

ind. hypothesis

Note that $A' = U \cdot A$ for some $U \geq 0$ and $b' = Ub$

Set $y^T := y'^T \cdot U \geq 0$

$\Rightarrow y^T A = (y'^T U) A = y'^T (UA) = y'^T A' = 0$

$y^T b = (y'^T U) b = y'^T (Ub) = y'^T b' < 0 \quad \square$

Proof Corollary 9.2

Existence of $y \Rightarrow \underbrace{y^T A}_{\geq 0} x = y^T b < 0$ not satisfied by any $x \geq 0$

└ $Ax = b$

Assume there is no $x \geq 0$ with $Ax = b$

$\Rightarrow$ the LP $\max\ 0^T x$ s.t. $Ax = b$, $x \geq 0$ is infeasible

dual is $\min\ y^T b$ s.t. $y^T A \geq 0^T$, y unrestricted

dual is feasible, $y = 0$ is a solution

$\Rightarrow$ dual is unbounded

$\Rightarrow$ y with $y^T A \geq 0^T$ and $y^T b < 0$ $\square$

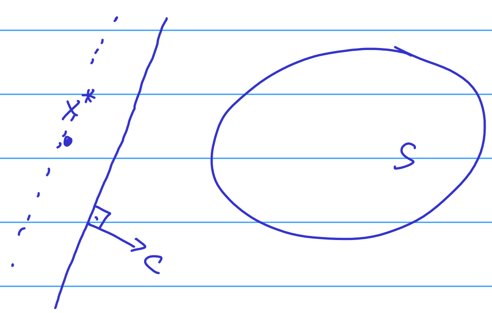
Separating Hyperplane Theorem

Another possible approach to Farkas' Lemma is the following separating hyperplane theorem:

Theorem 9.3.

Let S be a non-empty, closed, convex subset of $\mathbb{R}^n$ and let $x^* \in \mathbb{R}^n$ be a vector that does not belong to S. Then there exists some vector $c \in \mathbb{R}^n$ such that $c^T \cdot x^* < c^T \cdot x$ for all $x \in S$.

Proof: . . . $\square$

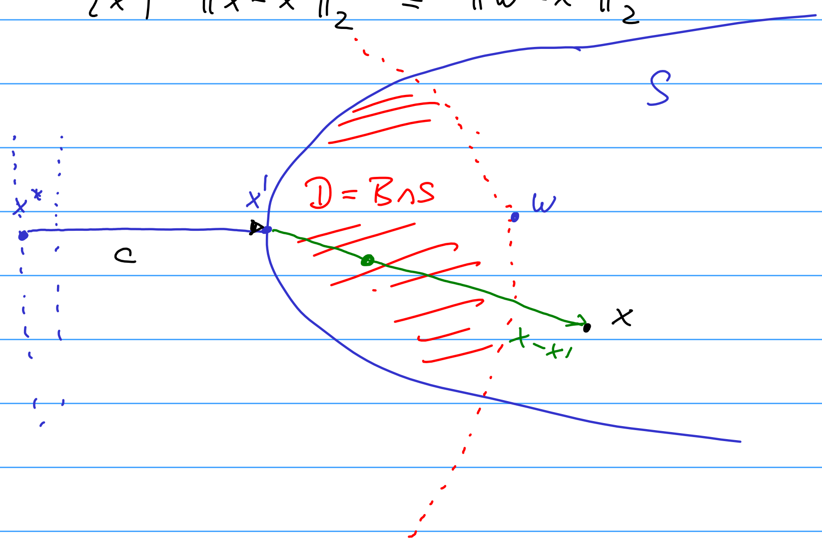


needs convexity

Proof Thm. 93

let $w \in S$ arbitrary

Set $B := \{x \mid \|x - x^*\|_2 \le \|w - x^*\|_2$



D is bounded, closed, $\neq \emptyset$

$\Rightarrow f(x) := \|x - x^*\|_2$ attains its minimum on D, say in x'

$\Rightarrow \|x' - x^*\|_2 \le \|x - x^*\|_2 \quad \forall x \in D$, even $\forall x \in S$

set $c := x' - x^*$

let $x \in S$ and $\lambda \in]0,1]$ with $\underbrace{x' + \lambda(x - x')}_{\circ} \in S$

$$\cancel{\|x' - x^*\|_2^2} \le \|x' + \lambda(x - x') - x^*\|_2^2$$

$$\le$$

$$\uparrow$$

$$\|a+b\|^2 = (a+b)^T(a+b) = \underbrace{a^Ta}_{\|a\|^2} + 2a^Tb + \underbrace{b^Tb}_{\|b\|^2}$$

$$= \cancel{\|x' - x^*\|^2} + 2\underbrace{\lambda(x' - x^*)^T}_{b}\underbrace{(x - x')}_{} + \lambda^2 \|x - x'\|^2$$

$$\Rightarrow \quad 0 \le 2\cancel{\lambda}(x' - x^*)^T(x - x') + \lambda^{\cancel{2}}\|x - x'\|^2$$

$$\lambda \to 0 \quad \Rightarrow \underbrace{(x' - x^*)^T}_{c^T}(x - x') \ge 0$$

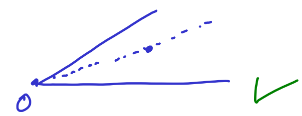
$$\Rightarrow c^T(x - x') \ge 0 \quad \Rightarrow c^T x \ge c^T x'$$

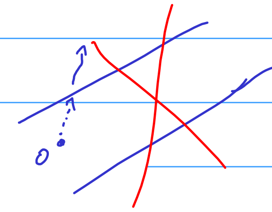
$\Rightarrow \quad c^T x \geq c^T x' = c^T x^* + c^T(\underbrace{x' - x^*}_{c})$

$\qquad\qquad\qquad\quad = c^T x^* + \underbrace{\|c\|^2}_{>0}$

$\Rightarrow \quad c^T x > c^T x^*$

$\in S$ arbitrary





Cones

Definition 9.4.

- **a** A set $C \subseteq \mathbb{R}^n$ is a cone if $\lambda \cdot x \in C$ for all $\lambda \geq 0$ and all $x \in C$.
- **b** A polyhedron of the form $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq 0\}$ is called polyhedral cone.

Remark: $0 \in P$ is the only possible vertex of a polyhedral cone P.
If $0 \in P$ is a vertex, then P is called pointed.

Theorem 9.5.

Let $C \subseteq \mathbb{R}^n$ be the polyhedral cone defined by the constraints ${a_i}^T \cdot x \geq 0$, $i = 1, \ldots, m$. Then, the following are equivalent:

- **i** The zero vector is an extreme point of C.
- **ii** The cone C does not contain a line.
- **iii** There exist n vectors out of the family $a_1, \ldots, a_m$, which are linearly independent.

Proof: Follows from Theorem 2.23. □