Technische Universität Berlin
Institut für Mathematik

Prof. Dr. Rolf H. Möhring
Torsten Gellert

ADM II
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Antje Lehmann

11. Exercise sheet

Due-date: Friday, 04.07.2014, *before* the lecture has started

This is the second last exercise sheet of the second half of the semester.

Exercise 61 **2+1 points**

Given a connected, undirected graph $G = (V, E)$ and edge-capacities $u_e > 0$ $(e \in E)$, a *legal ordering* of G is an ordering $V = \{v_1, v_2, \dots, v_n\}$ such that, where $V_i := \{v_1, \dots, v_i\}$,

$$u(\delta(V_{i-1}) \cap \delta(v_i)) \geq u(\delta(V_{i-1}) \cap \delta(v_j)) \quad \forall\ 2 \leq i < j \leq n.$$

(a) Describe an algorithm that finds a legal ordering in G. What bound for the running time can you guarantee?

(b) Find a legal ordering of graph G as illustrated in Exercise 66, starting with $v_1 = r$.

Exercise 62 **2+3+3 points**

Given a connected, undirected graph $G = (V, E)$ and edge-capacities $u_e > 0$ $(e \in E)$,

(a) Show that for any triple $u, v, w \in V$ holds $\mu(G; u, v) \geq \min\{\mu(G; w, u), \mu(G; w, v)\}$.

(b) Show that $u(\delta(v_n)) = \mu(G; v_n, v_{n-1})$ holds for any legal ordering $\{v_1, v_2, \dots, v_n\}$ of the nodes in G.

(c) Describe an algorithm to find a global min cut in time $\mathcal{O}(|V|^3)$ that does not need any maximum flow computation.

See Exercise 66 b) for a definition of $\mu(G; u, v)$.

Exercise 63 **4 points**

Let $G = (V, E)$ be an undirected graph with edge costs $c \in \mathbb{R}^E$ and $v_1 \in V$. Show that Dantzig, Fulkerson, and Johnson's relaxation of the TSP (see Lecture, LP 15.1) is equivalent to

$$\begin{array}{llcll} \min & \sum_{e\in E} c_e x_e & & & \\ \text{s.t.} & x(\delta(v)) & = & 2 & \forall v \in V \\ & x(\gamma(S)) & \leq & |S| - 1 & \forall S \subseteq V, v_1 \notin S \\ & 0 \leq & x_e & \leq 1 & \forall e \in E \end{array} \tag{1}$$

Exercise 64 **5 points**

Let $G = (V, E)$ be an undirected graph with edge costs $c \in \mathbb{R}^E$.

(a) Show how to solve the linear program

$$\begin{array}{llcll} \min & \sum_{e\in E} c_e x_e & & & \\ \text{s.t.} & x(\delta(v)) & = & 2 & \forall v \in V \\ & 0 \leq & x_e & \leq 1 & \forall e \in E \end{array}$$

as a minimum-cost flow problem.

(b) Use your construction to prove that there exists an optimal solution for which all variables have value 0, $\frac{1}{2}$, or 1.

Exercise 65 **Tutorial session – 0 points**

Given a asymmetric TSP as directed graph $G = (V, A)$ and $c\colon A \to \mathbb{R}^+$, i.e., $c(u, v) \neq c(v, u)$ for some $u, v \in V$.

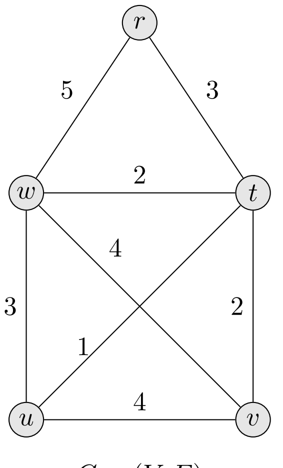
Describe how to transform the instance into a symmetric TSP such that tours in one instance correspond to tour in the other instance with the same length.

Exercise 66 **Tutorial session – 0 points**

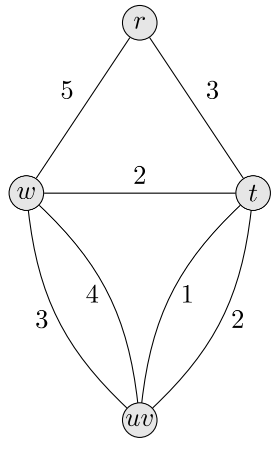
Given a connected, undirected graph $G = (V, E)$ and edge-capacities $u_e > 0$ ($e \in E$), the *(global) minimum cut problem* asks for an edge-set $A \subseteq E$ such that $\emptyset \subset S \subset V$ and $u(A)$ is minimum.

(a) Show that the global min cut problem in G can be solved by solving $|V| - 1$ maximum flow problems. What is the runtime of this algorithm?

(b) Let $\mu(G)$ denote the capacity of a minimum global cut in G. Moreover, given any pair $u, v \in V$, let $\mu(G; v, w)$ denote the capacity of a minimum (u, v)-cut in G, and G_{uv} be the graph obtained by identifying the nodes u and w to a single nodes (see figure below).
Show that $\mu(G) = \min\{\mu(G_{uv}), \mu(G; u, v)\}$.



$G = (V, E)$



G_{uv}

(c) Describe a faster algorithm to find a global min cut than the one proposed in a).