Technische Universität Berlin
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ADM II
Summer Term 2014

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6. Exercise sheet

Due-date: Friday, 30.05.2014, *before* the lecture has started

This is the last exercise sheet in the first half of the semester.

Exercise 26 **4 points**

Prove Birkhoff's Theorem by transforming the minimum-weight perfect matching problem in a bipartite graph to a minimum-cost flow problem.

Exercise 27 **4 points**

Let $n \in \mathbb{Z}_{>0}$. A matrix $A \in [0,1]^{n \times n}$ is *doubly stochastic* if the entries in each row and in each column sum up to 1. Show that each doubly stochastic matrix is a convex combination of permutation matrices.

Exercise 28 **4 points**

Let $G = (V, E)$ be an undirected graph and $c \in \mathbb{R}^E$. Consider the linear programming relaxation of the minimum-weight perfect matching problem:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & x\big(\delta(v)\big) = 1 \qquad \text{for all } v \in V \\ & x \geq 0 \end{aligned}$$

Show that there exists a feasible LP solution if and only if there exists a subset of edges and a set of odd circuits in G such that every node of G is a node of exactly one of the odd circuits or is incident to exactly one of the edges, but not both.

Exercise 29 **4 points**

Prove that the optimal value of the following problem is the minimum weight of a perfect matching of G.

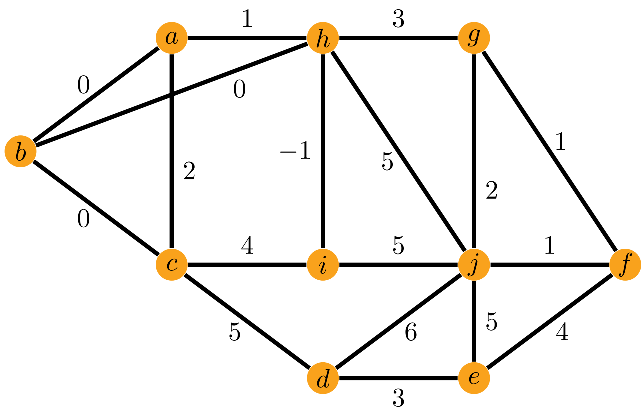
$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & x\big(\delta(v)\big) = 1 && \text{for all } v \in V \\ & x\big(\gamma(S)\big) \leq (|S| - 1)/2 && \text{for all } S \subseteq V,\ |S| \geq 3,\ |S| \text{ odd} \\ & x \geq 0 \end{aligned}$$

Exercise 30 **4 points**

Let M be a perfect matching of G and define the "cost" of an M-alternating circuit C of G to be $c\big(E(C) \setminus M\big) - c\big(E(C) \cap M\big)$. Prove that M is of minimum weight with respect to c if and only if there is no M-alternating circuit of negative cost.

Exercise 31 **Tutorial session – 0 points**

Find a minimum-weight perfect matching and an optimal solution of the dual problem for the following weighted graph:



Exercise 32 **Tutorial session – 0 points**

Let $G = (V, E)$ be an undirected graph, let $c \in \mathbb{R}^E$, and let k be a positive integer. Show how the problem of finding a minimum-weight matching having cardinality k can be reduced to a minimum-weight perfect matching problem.

Exercise 33 **Tutorial session – 0 points**

Show that there exists a non-bipartite graph with the property that, for every $c \in \mathbb{R}^E$, the optimal value of the linear program in Exercise is equal to the minimum weight of a perfect matching of G.

Exercise 34 **Tutorial session – 0 points**

Let $G = (V, E)$ be an undirected graph. Prove that G has a perfect matching if and only if there is an $x \in \mathbb{R}^E$ with:

$$\begin{aligned} x\big(\delta(v)\big) &= 1 && \text{for all } v \in V \\ x\big(\delta(S)\big) &\geq 1 && \text{for all } S \subseteq V,\ |S| \text{ odd} \\ x &\geq 0 \end{aligned}$$