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ADM II
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3. Exercise sheet

Due-date: Friday, 09.05.2014, *before* the lecture has started

Exercise 10 **3+3+2 points**

Consider the following variation of the diet problem: There are n foods and m nutrients where one unit of food j contains $a_{ij} \geq 0$ units of nutrient i. Consider a parent with two children with minimal nutritional requirements $b^1 \in \mathbb{R}^m_{\geq 0}$ and $b^2 \in \mathbb{R}^m_{\geq 0}$, respectively. Finally, let $c_j > 0$ be the cost of one unit of food j.

The parent has to buy food to satisfy the children's needs, at minimum cost. To avoid jealousy, there is the additional constraint that the amount to be spent for each child is the same.

(a) Provide a standard form formulation of this problem. What are the dimensions of the constraint matrix?

(b) If the Dantzig-Wolfe method is used to solve the problem in part a), construct the subproblems solved during a typical iteration of the master problem.

(c) Suggest a direct approach for solving this problem based on the solution of two single-child diet problems.

Exercise 11 **0+4 points**

Consider a digraph $D = (V, A)$ with a source node s and a sink node t.

(a) **(Tutorial session)** Consider the set of all s-t-flows $P := \{x \in \mathbb{R}^A \mid x \text{ is an } s\text{-}t\text{-flow}\}$. Give a complete characterization of the extreme points and the extreme rays of P (with proofs!).

(b) For a given flow value $d > 0$ consider the set of all s-t-flows of value d, i. e., $Q := \{x \in \mathbb{R}^A \mid x \text{ is an } s\text{-}t\text{-flow of value } d\}$. Give a complete characterization of the extreme points and the extreme rays of Q (with proofs!).

Hint: Use the flow decomposition theorem (Theorem 6.14).

Exercise 12 **3+3+2 points**

The *Maximum Multi-Commodity Flow Problem* is defined as follows:

Given: Digraph $D = (V, A)$ with capacities $u : A \to \mathbb{R}_{\geq 0}$ and k commodities given by source-sink pairs $(s_i, t_i) \in V \times V$, for $i = 1, \ldots, k$.

Task: Find s_i-t_i-flows x_i, for $i = 1, \ldots, k$, obeying arc capacities $\sum_{i=1}^k x_{i,a} \leq u(a)$, for every arc $a \in A$, such that the sum of the flow values $\sum_{i=1}^k \mathrm{ex}_{x_i}(t_i)$ is maximal.

(a) Formulate the Maximum Multi-Commodity Flow Problem as a linear program in arc variables.

(b) Apply Dantzig-Wolfe decomposition where the arc capacity constraints are the only "coupling constraints". Determine the extreme points and extreme rays of the polyhedra corresponding to the subproblems and formulate the master problem.

(c) Discuss how the pricing problem can be solved efficiently.

Exercise 13 **Tutorial session – 0 points**

Consider the following linear programming problem:

$$
\begin{array}{lrcrcrcrcrcrcr}
\text{maximize} & & & x_{12} & & & & & + & x_{22} & + & x_{23} & & \\
\text{subject to} & x_{11} & + & x_{12} & + & x_{13} & & & & & & & = & 20 \\
 & & & & & & & x_{21} & + & x_{22} & + & x_{23} & = & 20 \\
 & -x_{11} & & & & & - & x_{21} & & & & & = & -20 \\
 & & - & x_{12} & & & & & - & x_{22} & & & = & -10 \\
 & & & & - & x_{13} & & & & & - & x_{23} & = & -10 \\
 & x_{11} & & & & & & & & & + & x_{23} & \leq & 15 \\
 & x_{ij} \geq 0 & & & & & & & & & & & &
\end{array}
$$

We wish to solve this problem using Dantzig-Wolfe decomposition, where the constraint $x_{11} + x_{23} \leq 15$ is the only "coupling" constraint and the remaining constraints define a single subproblem.

(a) Consider the following two feasible solutions for the subproblem:

$$\mathbf{x}^1 = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}) = (20, 0, 0, 0, 10, 10),$$

and

$$\mathbf{x}^2 = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}) = (0, 10, 10, 20, 0, 0).$$

Construct a restricted master problem in which $\mathbf{x}$ is constrained to be a convex combination of $\mathbf{x^1}$ and $\mathbf{x^2}$. Find the optimal solution and the optimal simplex multipliers for the restricted master problem.

(b) Using the simplex multipliers calculated in part (a), formulate the subproblem and solve it by inspection.

(c) What is the reduced cost of the variable λ_i associated with the optimal extreme point $\mathbf{x}^i$ obtained from the subproblem solved in part (b)?

(d) Compute an upper bound on the optimal cost.