

2. Exercise sheet

Due-date: Friday, 02.05.2014, *before* the lecture has started

Exercise 4

6 points

Let $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = m$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Consider the linear program in standard form

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

and assume that the columns A_1, \dots, A_m of A form an optimal basis. Now the first column A_1 is changed to $A_1 + \delta \cdot A_0$ for some $A_0 \in \mathbb{R}^m$ and $\delta \in \mathbb{R}$. Consider the matrix $B(\delta) \in \mathbb{R}^{m \times m}$ consisting of the columns $A_1 + \delta \cdot A_0, A_2, \dots, A_m$. Let $\delta_1 < 0 < \delta_2$ such that $B(\delta)$ is non-singular for all $\delta_1 \leq \delta \leq \delta_2$. Show that the subset of the interval $[\delta_1, \delta_2]$ for which $B(\delta)$ is an optimal basis is again a closed interval.

Exercise 5

2+2+2 points

For fixed $\theta \in \mathbb{R}_{\geq 0}$ consider the linear programming problem

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 = \theta \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Find an optimal solution for all values of θ .
- Draw a graph showing the optimal cost as a function of θ .
- Use the picture in b) to obtain the set of all dual optimal solutions, for every value of θ .

Exercise 6

3+2+3 points

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c, d \in \mathbb{R}^n \setminus \{0\}$. For $\theta \in \mathbb{R}$ consider the following parametric program:

$$\begin{aligned} g(\theta) := \min \quad & (c + \theta \cdot d)^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

- Let $\theta_1 < \theta_2$ and suppose that $g(\theta)$ is linear for $\theta \in [\theta_1, \theta_2]$. Is it true that there exists a unique optimal solution when $\theta_1 < \theta < \theta_2$? Prove or provide a counterexample.
- Suppose that for some value of θ , there are exactly two distinct basic feasible solutions that are optimal. Is it true that they must be adjacent? Prove or provide a counterexample.
- Let θ^* be a breakpoint of the function $g(\theta)$. Let x^1, x^2, x^3 be pairwise different basic feasible solutions, all of which are optimal for $\theta = \theta^*$. Suppose that x^1 is a unique optimal solution for $\theta < \theta^*$, x^3 is a unique optimal solution for $\theta > \theta^*$, and x^1, x^2, x^3 are the only optimal basic feasible solutions for $\theta = \theta^*$. Provide an example to show that x^1 and x^3 need not be adjacent.

Exercise 7**Tutorial session – 0 points**

For $\theta \in \mathbb{R}$, consider the following linear program:

$$\begin{array}{llll} \min & 4x_1 & + 5x_3 & \\ \text{s.t.} & 2x_1 + x_2 - 5x_3 & & = 1 - 2\theta \\ & -3x_1 & + 4x_3 + x_4 & = 2 - 3\theta \\ & & & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

- For fixed $\theta := 0$, write down a simplex tableau and find an optimal solution. Is it unique?
- For fixed $\theta := 0$, write down the dual problem and find an optimal solution. Is it unique?
- Find an optimal solution and the value of the optimal cost as a function of $\theta \in \mathbb{R}$.

Exercise 8**Tutorial session – 0 points**

For $\delta \in \mathbb{R}$ consider the following linear program:

$$\begin{array}{llll} \min & -5x_1 - x_2 + 12x_3 & & \\ \text{s.t.} & (3 + \delta)x_1 + 2x_2 + x_3 & & = 10 \\ & 5x_1 + 3x_2 & + x_4 & = 16 \\ & & & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

For $\delta := 0$ set up a simplex tableau and solve the linear program (observe that x_1 and x_2 are the basic variables in an optimal solution). Let us keep x_1 and x_2 as the basic variables and let $B(\delta)$ be the corresponding basis matrix, as a function of δ (for small values of δ).

- Compute $B(\delta)^{-1} \cdot b$. For which values of δ is $B(\delta)$ a feasible basis?
- Compute $c_B^T \cdot B(\delta)^{-1}$. For which values of δ is $B(\delta)$ an optimal basis?
- Determine the optimal cost, as a function of δ , when δ is restricted to those values for which $B(\delta)$ is an optimal basis matrix.

Exercise 9**Tutorial session – 0 points**

Let $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Consider the following linear program in standard form:

$$\begin{array}{ll} \min & c^T \cdot x \\ \text{s.t.} & A \cdot x = b \\ & x \geq 0 \end{array}$$

Suppose that the first m columns A_1, \dots, A_m of A form an optimal basis $B \in \mathbb{R}^{m \times m}$ that leads to a non-degenerate optimal solution x^* , and a non-degenerate dual optimal solution p . We now change the first entry a_{11} of A_1 to $a_{11} + \delta$, for some $\delta \in \mathbb{R}$. Let $E \in \mathbb{R}^{m \times m}$ with entries all zero except for the top left entry e_{11} which is equal to 1.

- Show that if δ is small enough, $B + \delta \cdot E$ is a basis matrix for the new problem.
- Show that under the basis $B + \delta \cdot E$, the vector x_B of basic variables in the new problem is equal to $(I + \delta \cdot B^{-1} \cdot E)^{-1} \cdot B^{-1} \cdot b$.
- Show that if δ is sufficiently small, $B + \delta \cdot E$ is an optimal basis for the new problem.