Exercise Session 9

25.06.14

- Ex. 47
- TSP

Prog. Exercise: • presentation: next week (during ex. session) and other dates
• Lists to sign in: tomorrow during lecture

Exercise 47:

TSP: find shortest tour in graph $G = (V,E)$ and cost $c: E \to \mathbb{R}^+$
- visit every vertex exactly <u>once</u>

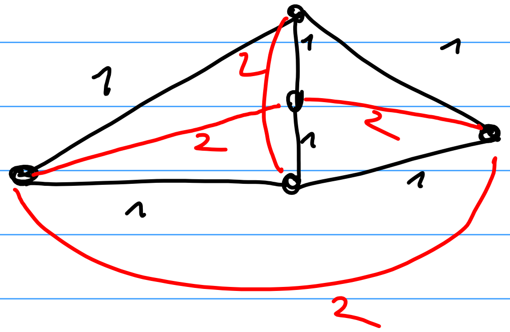
Decision problem of TSP is NP-complete: $\Rightarrow$ (probl.) no polyn. time algo.

Approximation: solution $ALG(I) \le \alpha \cdot OPT(I)$, OPT is opt. sol.
ALG is sol. of algorithm
I is an instance

- general case: no constant approximation possible ($\rightsquigarrow$ could decide Hamilton circuit)
- metric case: 3/2 approximation via Christofides
  Δ-inequality
- euclidean case: vertices are points in $\mathbb{R}^2$, $c_{uv} = \sqrt{(u_x - v_x)^2 + (u_y - v_y)^2}$
  $1+\varepsilon$ for every $\varepsilon > 0$, Running time is $\mathcal{O}(n^{1/\varepsilon})$

mostly: - complete graph
- symetric costs,

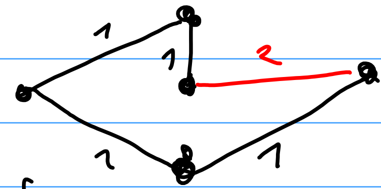
example:



TSP? ⇝ no tour, graph not hamiltonian

⇒ add uv edges as shortest u-v paths

TSP:



Bounds: - measurement for the quality of tours

- 1-Tree:
  - cheapest edge for fixed vertex
  - MST on remaining graph,

  ⇝ improvement via punishment for wrong degree
- LP bounds

LP:

$$\min \sum_{e} c_e \cdot x_e$$

⇒ many variables ($O(n^2)$), n are used in tour

$$\sum_{e \in \delta(v)} x_e = 2 \quad \forall v \in V$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad \forall\, \emptyset \neq S \subsetneq V$$

⇒ expon. many constraints

$$x_e \geq 0$$

subtour elimination constraints (SEC)

- constraints:
  - separation via min s-t cuts
  - solve LP, add new constraint
- variables:
  - column generation

<u>general procedure:</u>
- solve LP with a few SEC
- generate new cuts and add them to the LP
- no cuts found ⇒ optimal solution
- fractional solution ⟶
  - add more cuts of another form (comb inequalities)
  - or: branch & bound

<u>bidirected LP:</u> (useable for undir. case as well)

$$\min \sum_{a \in A} c_a \cdot x_a$$

$$\sum_{a \in \delta^-(v)} x_a = 1 \qquad \text{for all } v \in V$$

$$\sum_{a \in \delta^+(v)} x_a = 1 \qquad \text{— " —}$$

SEC1: $\sum_{a \in \delta^+(S)} x_a \geq 1 \quad \forall\ \emptyset \neq S \subsetneq V$ (not needed for $\delta^-(S)$; replace S by $V \setminus S$)

SEC2: $\tau_s = 0$

$1 \leq \tau_v \leq n-1 \qquad \forall\, v \in V \setminus \{s\}$

$\tau_v \geq \tau_u + 1$ (if $x_{uv} = 1$) $\qquad \forall\, u \in V,\ v \in V \setminus \{s\}$

$\rightsquigarrow\ \tau_v \geq \tau_u + 1 - (1 - x_{uv}) \cdot n$

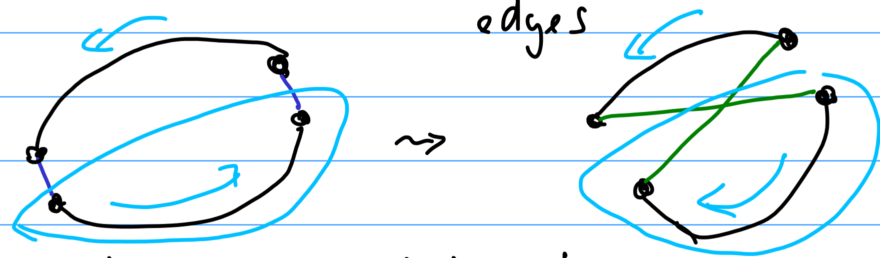
⇒ number of constr. is polyn., in practise BAD running time

no separation needed

Heuristics:

K-optimal tours: no replacement of K edges generates better solution

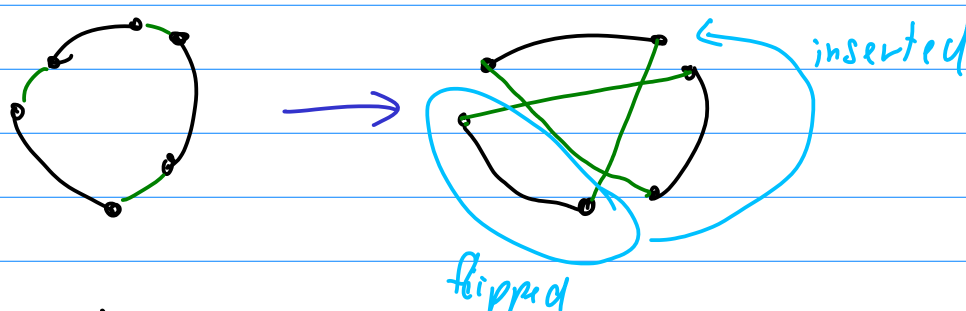
2-opt: • delete 2 edges, connect the path with two different edges



- check all possible neighbors in $O(n^2)$

- can be seen as flip() operation, ~~tour~~ as permutation, flip a subsequence

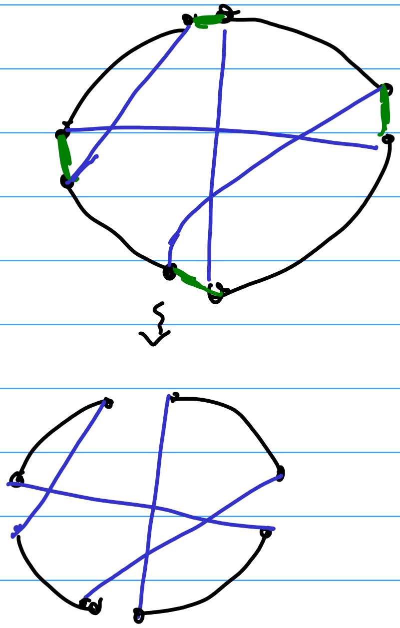
3 opt:



- can be seen as flip() and insert() combination

Observation: for every tour, there is a $K \leq n$ such that the opt. tour can be found by a local search on the K-neighborhood

Sequential k-opt move:



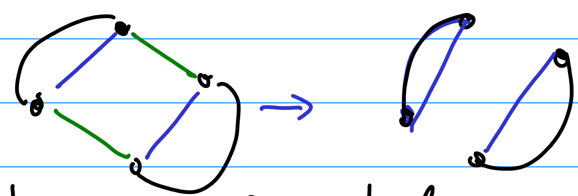
- alternating path describes change to new tour

$$E(T) \Delta E(P) = E(T')$$

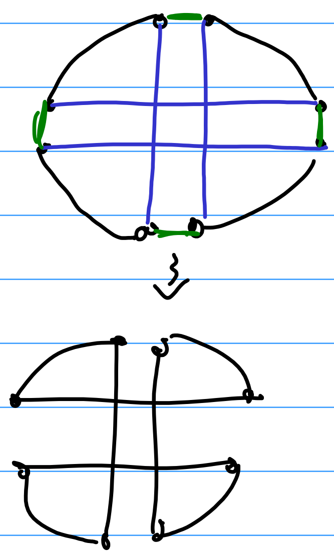
tour path new tour

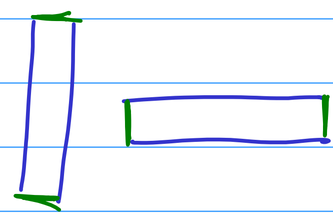
$\Rightarrow$ P is alternating path, s.t. $E(T) \Delta E(P)$ is a tour

- 2 problems:
  - not every path is valid:



  - every 2/3 opt move is sequential, but not every 4-opt move: "Double Bridge Move"





$\rightarrow$ 2 alternating path

History: Croes '58: 2opt as local search

Lin '65: 3-opt ⇒ much better results than 2-opt (4opt is to expensive to calculate)

Lin-Kernighan '73: adaptive k-opt, special sequences of flip/insert operations/alternating path

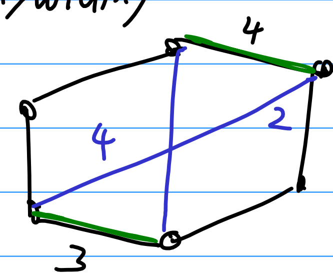
⇒ LK(+H) state-of-the-art heuristic for TSP, Concorde (exact solver) starts with LKH solution and adds more LP cuts and branch & bound

Lin Kernighan:

- find alternating path with positive gain g_t

  $$g_t := \sum_{i=0} (-1)^i \cdot c(e_i)$$

- use backtracking to find all such paths (→ add search depth, width)
- select best tour



$g_t = 4-2+3-4$
$= 1$

⇒ new tour shorter by 1