Exercise Session

18.06.14.

- send me an email if you would like to see the solution to a certain exercise in the ex. session

Gomory-Cuts:

Reminder: P is a polytope

- ILP over P = LP over P_I ($P_I :=$ convex hull of the integral points in P)
- P has only integral bfs $\Leftrightarrow$ $P = P_I$

$\Rightarrow$ we want to find P_I (but that's hard)

Gomory-Chvátal: $Ax \le b$, $x \in \mathbb{Z}^n$

- cut: $y^T A x \le \lfloor y^T b \rfloor$ for pos. linear combination y, with $y^T A \in \mathbb{Z}^m$, $y^T b$ not integral, (even: $y \in [0,1]^m$)

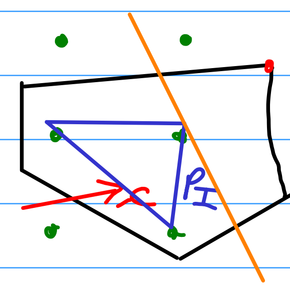
- $P^{(1)} := \bigcap_{\substack{y^T A \text{ integer} \\ y \ge 0}} \{ x \in P \mid y^T A x \le \lfloor y^T b \rfloor \}$ is the first chvátal closure

$P \supseteq P^{(1)} \supseteq P^{(2)} \supseteq \dots \supseteq P^{(k)} = P_I$ $\quad k :=$ chvátal rank of P

$\rightarrow$ chvátal closure is not easy to obtain

- often P_I not needed to solve ILP

$\rightarrow$ just cut fract. opt. sol. away



opt. sol of an LP

cut would suffice to solve ILP as LP

Cutting plane procedure:

Loop {

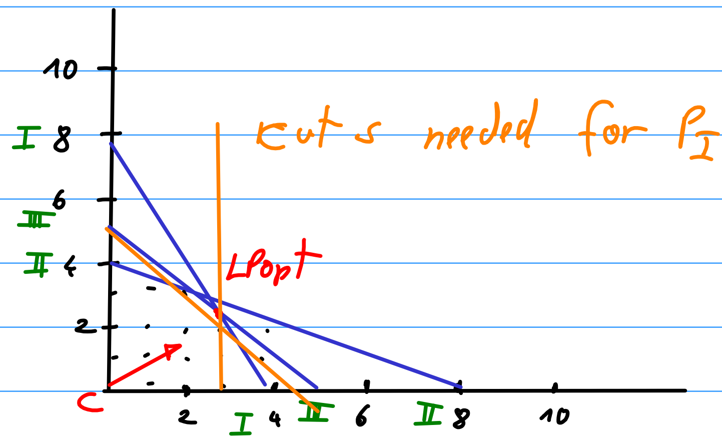
- calculate opt. solution x^* of the LP relaxation of P
- if x^* is integral, return x^*
- find separating hyperplane $w^T x \leq t$ with $\boxed{w^T x^* > t}$
- $P := \{x \in P \mid w^T x \leq t\}$

}

How to find a hyperplane? (Optimization is Separation)

$\rightarrow$ Gomory cuts give such a hyperplane

example:



$$\max \quad 3x_1 + 2x_2$$

$$\begin{aligned} \text{st.} \quad 4x_1 + 2x_2 &\leq 15 \quad \mathrm{I} \\ x_1 + 2x_2 &\leq 8 \quad \mathrm{II} \\ x_1 + x_2 &\leq 5 \quad \mathrm{III} \\ \Big(-x_1 &\leq 0 \\ -x_2 &\leq 0 \Big) \end{aligned}$$

LP opt: $x_1 = x_2 = 2.5$

obj. $= 12.5$

cut: $\frac{1}{2}\mathrm{I} + \mathrm{III}$

$\Rightarrow \underbrace{3x_1 + 2x_2}_{\text{cost function}} \leq \lfloor 12.5 \rfloor = 12$

$\rightarrow$ new LP opt: $x_1 = 2$

$x_2 = 3$

obj $= 12$

in general:

I is not a good constraint:

$\frac{1}{2}\mathrm{I} \Rightarrow 2x_1 + x_2 \leq \lfloor 7.5 \rfloor = 7$

general procedure:

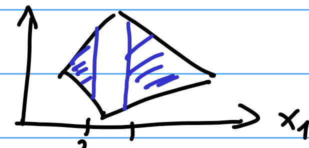
$a_1 x_1 + \dots a_n x_n \leq b$ $\quad$ a_j are integers

$\leadsto g = \gcd(a_1, \dots, a_n)$

$$\frac{a_1}{g} \cdot x_1 + \dots \frac{a_n}{g} x_n \leq \left\lfloor \frac{b}{g} \right\rfloor \quad \text{is a valid constraint}$$

$\leadsto$ assume gcd of $a_1, \dots, a_n$ is 1 for every constraint

- other techniques: disjunctive constraints
  - starting with (P)
  - branching: e.g., $x_1 \leq 2$ and $x_1 \geq 3$



$2 \cdot \mathrm{III} \Rightarrow 2x_1 + 2x_2 \leq 10 \iff 3x_1 + 2x_2 + (2 - x_1) \leq 12$

$\mathrm{I} \Rightarrow 4x_1 + 2x_2 \leq 15 \iff 3x_1 + 2x_2 + (x_1 - 3) \leq 12$

2 cases:

- $x_1 \leq 2$: $\rightarrow$ $\underline{3x_1 + 2x_2} \leq 3x_1 + 2x_2 + \underbrace{(2 - x_1)}_{\geq 0} \underline{\leq 12}$
- $x_1 \geq 3$: $\rightarrow$ $\underline{3x_1 + 2x_2} \leq 3x_1 + 2x_2 + \underbrace{(x_1 - 3)}_{\geq 0} \underline{\leq 12}$

$\Rightarrow$ in both cases $3x_1 + 2x_2 \leq 12$

enough to solve IP with an LP solver

in general: $k \in \mathbb{Z}$, $\alpha, \beta \geq 0$ $\quad$ (cases $x_j \geq k+1$ or $x_j \leq k$)

we have: $\sum_j a_j x_j + \alpha(x_j - (k+1)) \leq b$

and $\sum_j a_j x_j + \beta(k - x_j) \leq b$

$\leadsto \boxed{\sum_j a_j x_j \leq b}$

modular arithmetic and Gomory cuts:

- based on equality constraints (obtained as pos. lin. combination)

$\sum_j a_j x_j = b$ with $x_j \geq 0$, x_j are integers

given some $d \in \mathbb{N}$, $r_j := a_j \mod d$
$s := b \mod d$

- x_j are integers $\Rightarrow$ $\sum_j r_j x_j \equiv s \mod d$
- because $s < d$ and $\sum_j r_j x_j \geq 0$

$\Rightarrow$ $\boxed{\sum_j r_j x_j \geq s}$ new constraint

- special case: $d = 1$

$$\boxed{\sum_j (a_j - \lfloor a_j \rfloor) x_j \geq b - \lfloor b \rfloor}$$

Gomory cut:

$\sum_j a_j x_j = b$
$+ \sum_j \lfloor a_j \rfloor x_j \leq \lfloor b \rfloor$

procedure: → solve the LP
→ choose row in the resulting tableau and add the Gomory cut for it

example: max $3x_1 + 2x_2$

$$\begin{aligned}
4x_1 + 2x_2 + s_1 \quad &= 15 \\
x_1 + 2x_2 \quad + s_2 \quad &= 8 \\
x_1 + x_2 \quad\quad + x_3 &= 5 \\
x_1, x_2 \geq 0 \quad & \\
\text{integer} \quad &
\end{aligned}$$

starting with bfs: $s_1 = 15$, $s_2 = 8$, $s_3 = 5$

- basis change: x_1 enters basis
s_1 leaves basis

• basis change: x_2 enters basis
s_3 leaves basis

tableau:

	x_1	x_2		s_2	
$-12,5$	0	0	$-1/2$	0	-1
$2,5$	1	0	$1/2$	0	-1
$0,5$	0	0	$1/2$	1	-3
$2,5$	0	1	$-1/2$	0	2

in another form max $12.5 - 1/2\, s_1 - s_3$ $\quad x_1, x_2, s_1, s_2, s_3 \geq 0$

$$x_1 = 2,5 - 1/2\, s_1 + s_3$$
$$s_2 = 0,5 - 0,5\, s_1 + 3 s_3$$
$$x_2 = 2,5 - 0,5\, s_1 - 2 s_3$$

sol: $s_1 = 0,5$
$x_1 = x_2 = 2,5$

→ Gomory cuts:

1. row $\quad x_1 + 0,5\, s_1 - s_3 = 2,5$
$\quad 0,5\, s_1 \geq 0,5 \iff \boxed{s_1 \geq 1}$ new cut

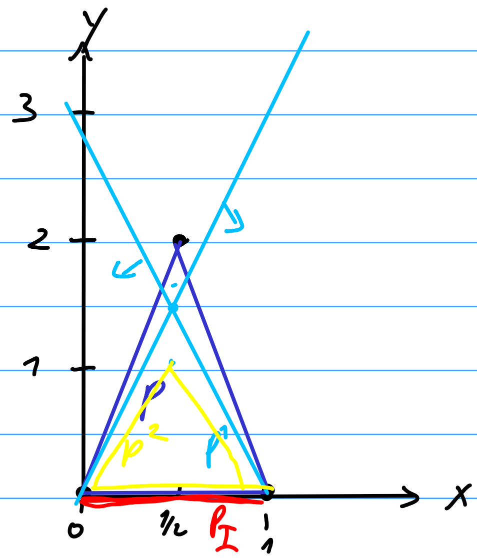
2. row: $\quad s_2 + 0,5\, s_1 - 3 s_3 = 0,5$
$\quad \to 0,5\, s_1 \geq 0,5 \iff s_1 \geq 1$

3. row: $\quad x_2 + 0,5 s_1 + 2 s_3 = 2,5$
$\quad \to 0,5 s_1 \geq 0,5 \iff s_1 \geq 1$

→ $s_1 \geq 1$ cuts away the optimal LP solution,
LP finds an integral opt. sol.

• $s_1 \geq 1$ gives: $4x_1 + 2x_2 \leq 14$ (from $4x_1 + 2x_2 + s_1 = 15$)
$\iff 2x_1 + x_2 \leq 7$

Chvátal Rank can be very high:
example: convex hull of $(0,0)$, $(1,0)$, $(1/2, 2)$



Constraints:

$$-y \le 0 \qquad\qquad -y \le 0$$
$$y \le 4x \quad \Leftrightarrow \quad y - 4x \le 0 \qquad \mathrm{I}$$
$$y \le -4(x-1) \qquad y + 4x \le 4 \qquad \mathrm{II}$$

$$(1/8)\mathrm{I}: \quad 1/8\, y - 1/2\, x \le 0$$
$$(7/8)\mathrm{II}: \quad 7/8\, y + 7/2\, x \le 7/2$$
$$\overline{y + 3x \le \lfloor 7/2 \rfloor = 3}$$

needed for P_I: $y \le 0$

analog: $(7/8)\mathrm{I} + (1/8)\mathrm{II}$

$\leadsto$ $y - 3x \le 0$

$\leadsto$ in fact:
$$-y \le 0$$
$$y - 3x \le 0 \qquad \text{is } P^{(1)}$$
$$y + 3x \le 3$$