Exercise Session 2 — 30.04.14

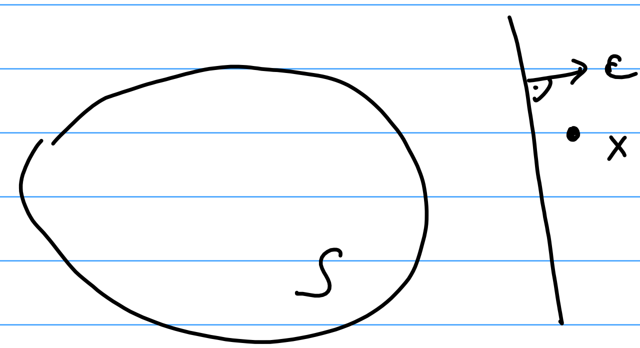
Topics: Separation

Next week:
- Wedn.: Lecture
- Thurs.: exercise session (intro. to programming exercise)
- Friday: Lecture

Separating Hyperplane:

Thm: set $S \subset \mathbb{R}^n$, $\emptyset \neq S$, S is closed and convex, $x^* \in \mathbb{R}^n$, $x^* \notin S$

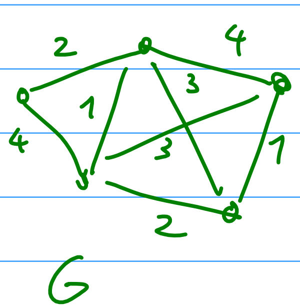
$\Rightarrow \exists c \in \mathbb{R}^n: \; c^T x^* < c^T x \quad \forall x \in S$



→ How to find this separating hyper-plane?

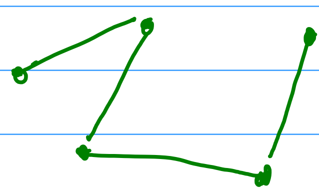
Example: Minimum Spanning Tree

MST: Given $G = (V, E)$, $c: E \to \mathbb{R}$
Find a cheapest tree T in G that spans all vertices



G

MST: (Kruskal)



LP formulation for MST:

$$\min \sum_{e \in E} c_e x_e$$

$x(E) = |V|-1$ → exactly $n-1$ edges

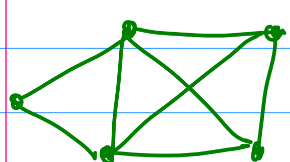
$x(\gamma(S)) \le |S|-1 \quad \forall\, \emptyset \ne S \subsetneq V$ → no subset contains "more" than $|S|-1$ edges

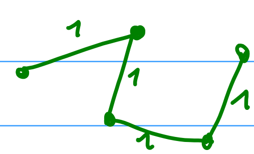
$x_e \cancel{\in \{0,1\}}$ ≥ 0

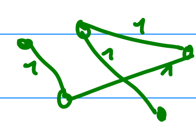
• for $S \subseteq V$ let $\underbrace{\gamma(S) := \{e = \{v,w\} \in E \mid v,w \in S\}}_{\text{all edges in set } S}$

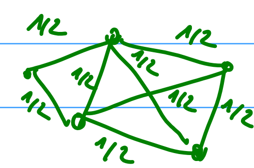
• for $x \in \mathbb{R}^{|E|}$ and $B \subseteq E$:
let $x(B) := \sum_{e \in B} x_e$ sum up vector x for set B

Notice: • LP defines polytope (MST polytope)
• extreme points of polytope are MSTs ($x_e \in \{0,1\}$)

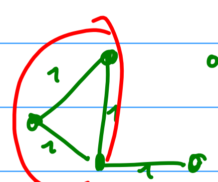


x_1  in MST polytope

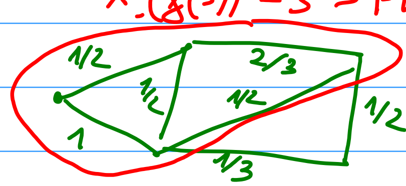
x_2  — " —

x_3  in MST polytope

$x_3 = \frac{1}{2} x_1 + \frac{1}{2} x_2$

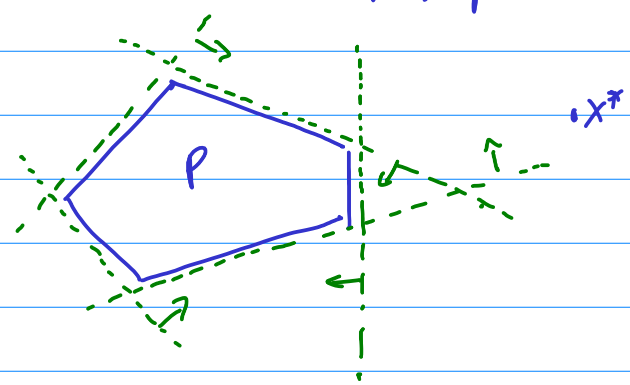
 not in MST polytope

$x(\gamma(S)) = 3 > |S|-1 = 3-1 = 2$

 not in MST polytope

$x(\gamma(S)) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{1}{2} > \underbrace{4-1}_{|S|-1}$

in general: consider polytope



- consider all facets (constraints)
- a shifted facet separates x^* and P

$\rightsquigarrow$ MST polytope: one constraint for every subset S
$\rightarrow$ expon. many constr.

(often constraints explicitly given)

more precisely: for a complete graph K_n there are n^{n-2} spann. trees, each an extreme point of the polytope

efficient way for separation:

define graph: $D=(V', A)$

- $V' := \{s,t\} \cup V \cup E$
- $A := \{(s,e) \mid e \in E\} \cup \{(e,v) \mid v \in V, e \in \delta(v)\} \cup \{(v,t) \mid v \in V\}$

for a given $x \in \mathbb{R}^E_{\geq 0}$ with $x(E) = |V|-1$ and $v' \in V$ define:

$D_{v'} := (V', A \cup \{s, v'\})$

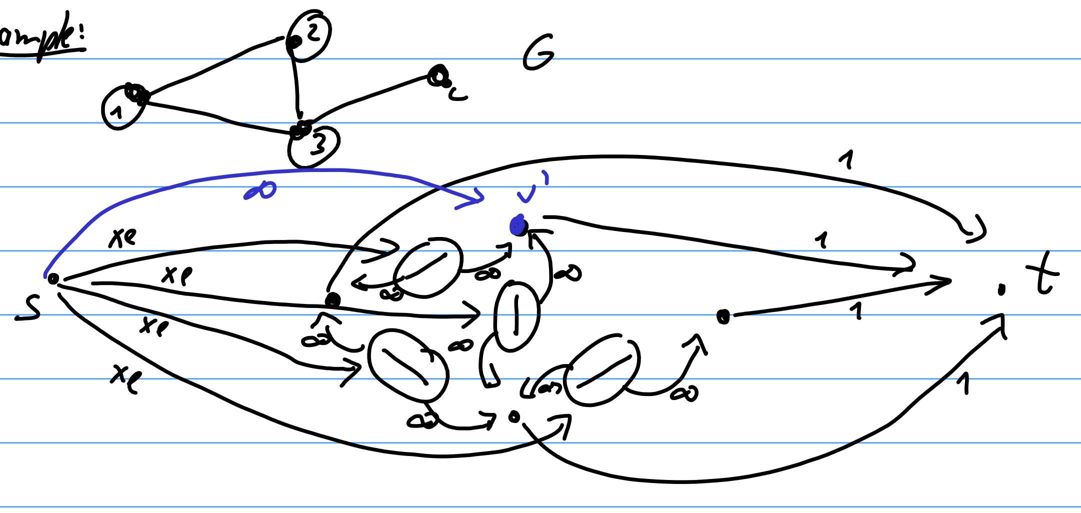
$u((s,e)) := x_e \quad \forall e \in E$

$u((e,v)) := \infty \quad \forall e \in \delta(v), v \in V$

$u((v,t)) := 1 \quad \forall v \in V$

$u((s,v')) := \infty$

example:

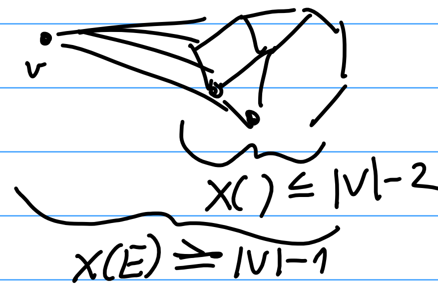


Thm: $x \in \mathbb{R}^E_{\geq 0}$, $x(E) = |V|-1$

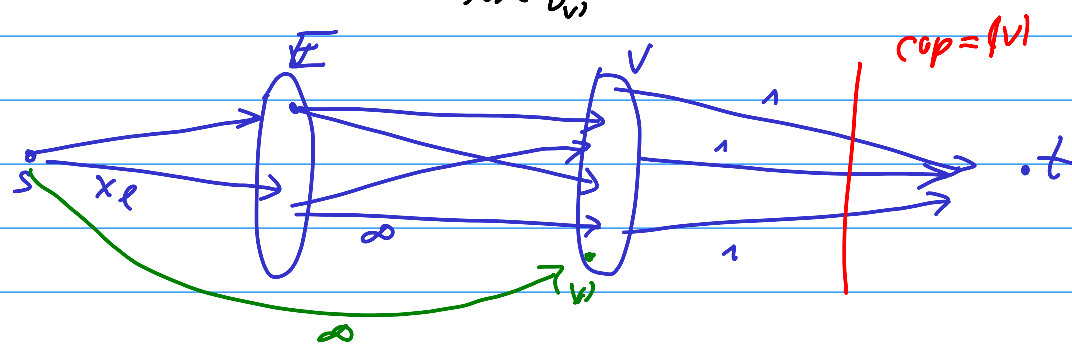
x fulfills LP constraints $\iff$ max s-t flow value in $D_{v'} = |V|$ for every $v' \in V$

Proof: "$\Rightarrow$" $x(\gamma(S)) \leq |S|-1 \quad \forall\ \emptyset \neq S \subsetneq V$, $x(E) = |V|-1$

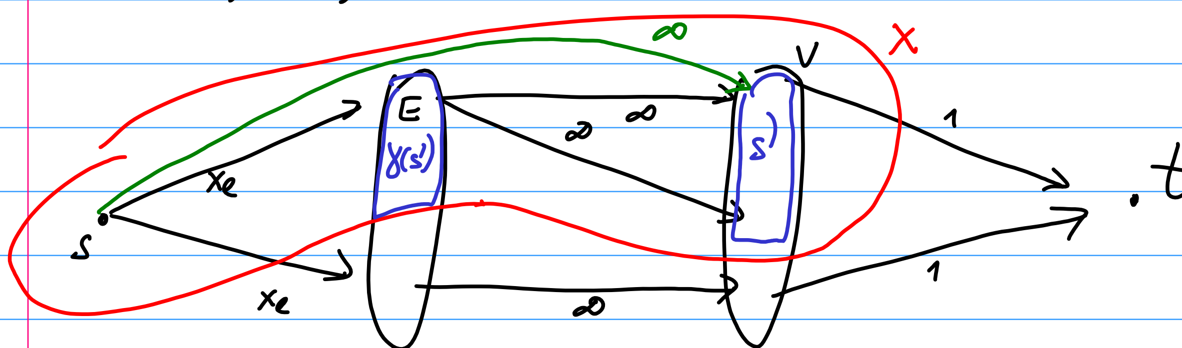
also: $x(\delta(v)) \geq 1$



- consider $D_{v'}$
  max flow value $\leq |V|$,
  because of cut $\sum_{(v,t) \in D_{v'}} u(e) = |V|$



- consider s-t cut $\delta(X)$ in $D_{v'}$ (arbitrary)
  $s \in X$, $t \notin X$, $X \subset \{s\} \cup E \cup V$



- assume: $v' \in X$, otherwise: $cap(\delta(X)) = \infty$
  assume: no edge (e,v) with $e \in E$ and $v \in V$ in cut, otherwise $cap(\delta(X)) = \infty$

define $s' := \{v \mid v \in X\}$

$$cap(\delta(X)) = |\{v \mid v \in X\}| + \sum_{e \notin X} x_e$$
$$= |s'| + (|V|-1) - \sum_{e \in X} x_e$$

$$= |S'| + |V| - 1 - \underbrace{x(\gamma(S'))}_{\leq |S|-1} \geq \cancel{|S'|} + |V| \cancel{-1} \cancel{-|S'|} \cancel{+1} = |V|$$

$\Rightarrow$ max flow value $\geq |V|$ for all $v \in V$

$\Rightarrow$ max flow value $= |V|$ for all $v \in V$ $\triangle$

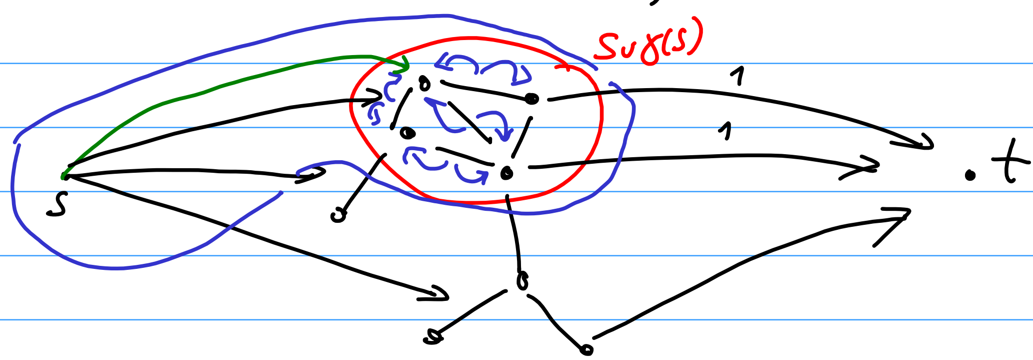
„$\Leftarrow$" $x \in \mathbb{R}^E_{\geq 0}$, $x(E) = |V| - 1$,

max flow value of $D_{v'} = |V|$ for every $v' \in V$

to show: $x(\gamma(S)) \leq |S| - 1 \quad \forall\ \emptyset \neq S \subsetneq V$

for arbitrary $S \subset V$:

choose set $X := \{s\} \cup S \cup \gamma(S)$, $v' \in S$



$$\text{cap}(\delta(X)) = \underbrace{|S|}_{\substack{\text{arcs } (v,t) \text{ with} \\ u(v,t)=1 \text{ for } v\in S}} + \underbrace{\sum_{e \notin \gamma(S)} x_e}_{\substack{(s,e) \text{ arcs} \\ \text{for } e \notin \gamma(S)}}$$

- no edge $\to$ vertex arc in $\delta(X)$
- (s, v') with $u = \infty$ not $\delta(X)$

$$= |S| + \sum_{e \notin \gamma(S)} x_e \geq |V|$$ flow condition

$$= |S| + x(E \setminus \gamma(S)) \geq |V|$$

$$= |S| + \cancel{x(E \setminus \gamma(S))} \geq x(E) + 1 = \cancel{x(E \setminus \gamma(S))} + x(\gamma(S)) + 1$$

$$\Leftrightarrow |S| - 1 \geq x(\gamma(S))$$ $\boxtimes$

MST separation can be done by n max s-t flow calculations

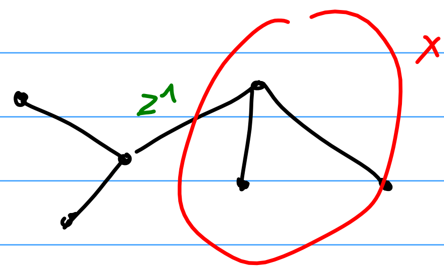
MST: Alternative Formulierung:

$$\min \sum_{e \in E} c_e \cdot x_e$$

$$\sum_{e \in \delta(X)} \geq 1 \quad \forall \emptyset \neq X \subsetneq V$$

$$x(E) = |V| - 1$$

$$x \geq 0$$



→ separation! • find minimal cut in graph
⇝ s-t flow for ever $s \in V$