1. Exercise Session

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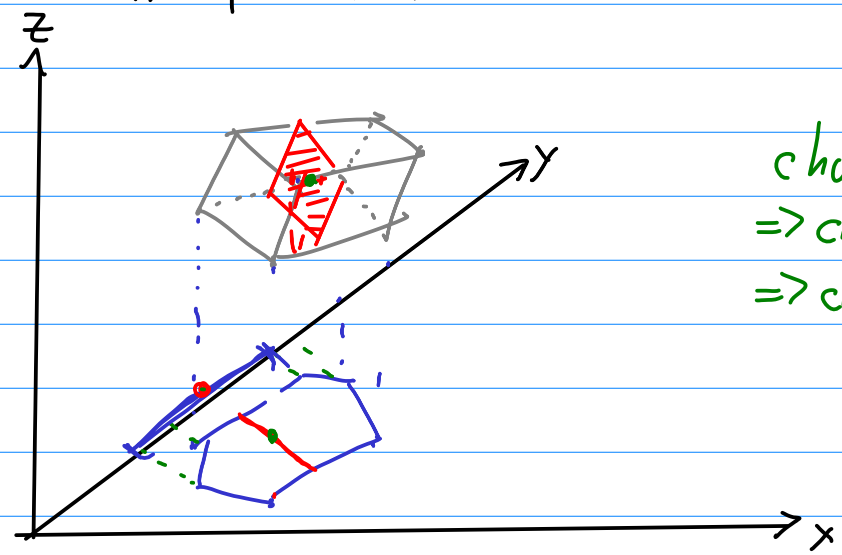
HA 0: noch keine Gruppe => Email mit Nr. 0

Fourier Motzkin Elimination:

- procedure to solve question $\exists x \in P$ for $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$
    - very inefficient: expon. running time
    - nice theoretical tool: derivation of the Farkas Lemma
- idea: project the polytope onto a smaller dimension to get rid of a variable



choose $y \in$ Interval
=> choice for x
=> choice for z

formula: • $Ax \leq b$ $\qquad x' := (x_2, \dots, x_n)$, $A' := (A_2 | \dots | A_n)$

$$Ax \leq b \iff \begin{aligned} x_1 + a_i' x' &\leq b_i & \forall i \in M^+ \\ -x_1 + a_i' x' &\leq b_i & \forall i \in M^- \\ a_i' x' &\leq b_i & \forall i \in M \end{aligned}$$

$$\iff \max_{i \in M^-} (a_i' x' - b_i) \leq x_1 \leq \min_{i \in M^+} (b_i - a_i' x')$$

$$a_i' x' \leq b_i \quad \forall i \in M$$

$$\Leftrightarrow \quad \cdot\, a_i' x' - b_i \le b_j - a_j' x' \qquad \forall\ i \in M^-, \forall j \in M^+$$
$$a_i' x' \le b_i \qquad \forall i \in M$$

$$\Leftrightarrow \quad \underbrace{\begin{matrix} (a_i' + a_j')\, x' \le (b_j - b_i) \\ a_i'\, x' \le b_i \end{matrix}}_{=:\, A'' x' \le b'} \qquad \begin{matrix} \forall\ i \in M^-, \forall j \in M^+ \\ \forall i \in M \end{matrix}$$

$\Rightarrow$ FME: • find $x' = x_2, \dots, x_n$ with $A'' x' \le b'$
• choose any value for x_1 from interval
$$\left[\max_{i \in M^-} (a_i' x' - b_i);\ \min_{i \in M^+} (b_i - a_i' x')\right]$$

• solving LP: $\min c^T x$, $Ax \le b$, $(x \ge 0)$ $\Rightarrow$ find (x, λ) s.t. $\tilde{A} x \le \tilde{b}$, $c^T x - \lambda \le 0$

· eliminate $x_1, \dots, x_n$
· choose λ as small as possible
$\rightsquigarrow c^T x = \lambda$

example: cube $1 \le x_i \le 2 \quad i \in [n]$ — M: $x_i \le 2 \quad \forall\ i \in \{2, \dots, n\}$

$x_i \le 2 \quad \forall i \in [n]$ $\rightsquigarrow$ $-x_i \le -1$

$-x_i \le -1 \quad \forall i \in [n]$

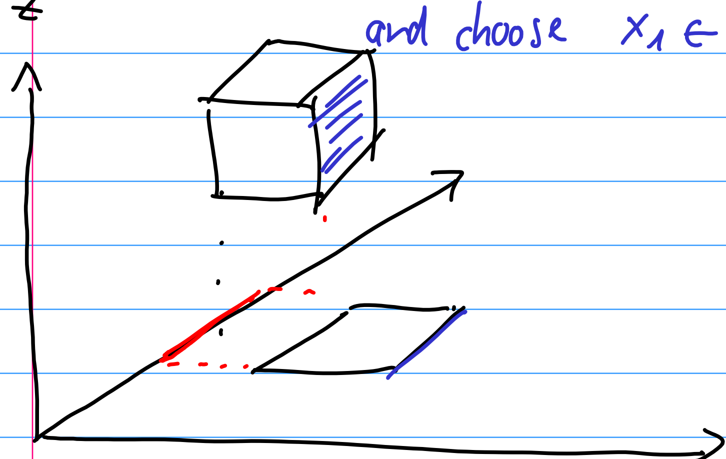
M^+: $x_1 \le 2$

M^-: $\underbrace{-x_1 \le -1}_{0 \le 1}$

$\rightsquigarrow$

find $x_2, \dots, x_n$: $x_i \le 2$, $-x_i \le -1$ $\quad \forall\ i \in \{2, \dots, n\}$

and choose $x_1 \in [1, 2]$



Farkas Lemma:

Farkas Lemma

$A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^n$. Exactly one of the following alternatives hold:

1. $\exists\ x \geq 0$ such that $Ax = b$ $\cdot\ P \neq \emptyset$
2. $\exists$ some vector p such that $p^T A \geq 0$ and $p^T b < 0$ $\cdot\ P = \emptyset$

$P := \{x \mid Ax = b\}$

$\rightsquigarrow$ p, x are certificate for answer if $P = \emptyset$?

$\Rightarrow LP \in NP,\ LP \in coNP$

(Farkas Lemma: Condition for existance of sol.)

Overview:

Simplex $\Rightarrow$ strong duality thm $\underset{(\Leftarrow)}{\Rightarrow}$ Farkas $\Leftarrow$ FME

Alternative Form:

Alternative Farkas Lemma

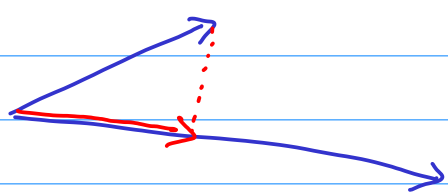
Let $a_1, \ldots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^n$. Following statements are equivalent.

1. for all $y \in \mathbb{R}^n$: $y^T a_i \geq 0 \forall i = 1, \ldots, m \Rightarrow y^T b \geq 0$

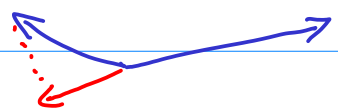
   for all y: (y has a non-neg. projection onto every a_i $\Rightarrow$ y has non-neg proj. onto b)

2. $b \in \text{Cone}(a_1, \ldots, a_m)$ b lies in the cone spanned by $a_1, \ldots, a_m$

non ng. projection:



neg. proj.:

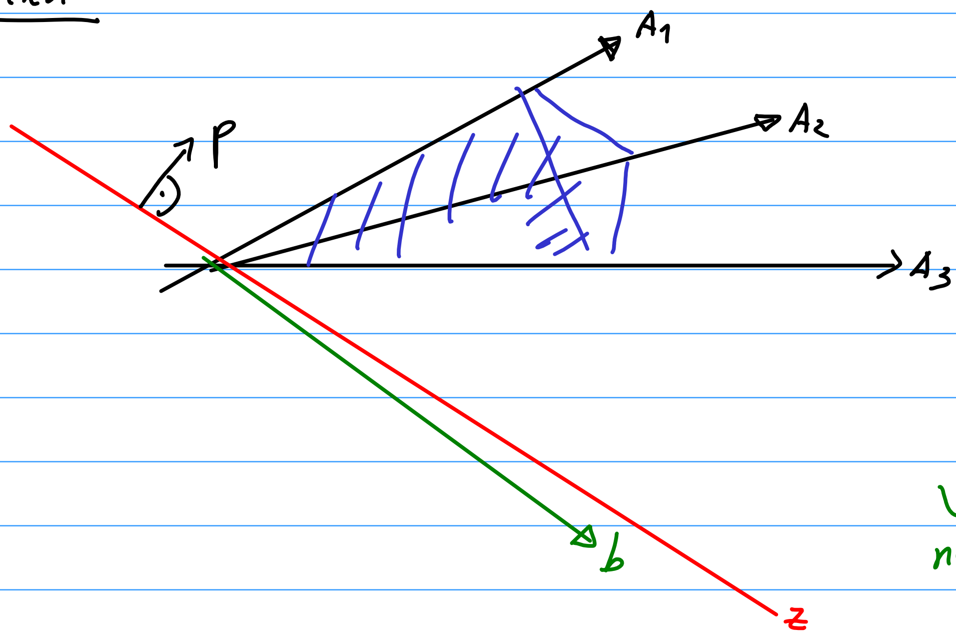


Reformulation: $\neg(1)$: $\exists y \in \mathbb{R}^n$: $y^T A \geq 0$ and $y^T b < 0$

$\neg(2)$: $Ax = b$, $x \geq 0$ is not solvable

(can not combine b as a pos. linear comb. of A_i)

Sketch:



- $b \notin \text{Cone}(A_1, A_2, A_3)$
- hyperplane $\{z \mid p^T z = 0\}$ separates b from A_i
- $p^T A \geq 0$ (non-neg. projection)
- $p^T b < 0$ (neg. proj.)

$\rightsquigarrow$ not possible if $b \in$ Cone

Farkas $\Rightarrow$ Strong Duality Theorem

strong. duality thm:

$$(P) \quad \min c^T x \quad Ax = b, \quad x \geq 0 \qquad\qquad (D) \quad \max y^T b \quad y^T A \leq c^T, \quad y \text{ free}$$

Thm: (P) has an opt. sol. if and only if (D) has an optimal solution and their obj. values coincide.

Proof: (via Farkas)

> **Theorem 9.1.**
> The system $A \cdot x \leq b$ has a solution x, if and only if there is no vector y satisfying $y \geq 0$, $y^T \cdot A = 0$ and $y^T \cdot b < 0$.

- y' is opt. solution of (D), $\delta := y'^T b$
- $\begin{array}{l} y^T A \leq c^T \\ -y^T b \leq -\delta - \varepsilon \end{array}$ has no solution for $\varepsilon > 0$

$\overset{\text{Farkas}}{\Rightarrow}$ $\exists \lambda \geq 0$: $\lambda^T \cdot \begin{pmatrix} -b \\ A \end{pmatrix} = 0$ and $\lambda^T \begin{pmatrix} -\delta-\varepsilon \\ c \end{pmatrix} < 0$

- $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda' \end{pmatrix}$ with $\lambda' \in \mathbb{R}^n$

assume: $\lambda_1 = 0$: $\lambda' \cdot A = 0$ and $\lambda' \cdot c < 0$,

remember: $A^T y' \leq c^T$ $\Rightarrow$ ⚡ Farkas $\Rightarrow \lambda_1 \neq 0$

$\Rightarrow \lambda_1 > 0$, scale $\begin{pmatrix} \lambda_1 \\ \lambda' \end{pmatrix}$ such that $\lambda_1 = 1$

$\underbrace{\lambda'^T \cdot A = b^T}_{\substack{A \cdot \lambda' = b, \\ \text{feas. sol. for } P}}$ and $\underbrace{\lambda'^T c < \delta + \varepsilon}_{c^T \lambda' < \delta + \varepsilon}$ $\Rightarrow$ opt. value for (P) is δ (via weak duality)

$\Rightarrow \square$