ADM I Ex. Sess. 3 31.10.13

· Fourier Motzkin Elimination

Joseph Fourier (1826), Theodore Motzkin (1936, rediscovered)

Polyhedra $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ given

Question: • $\exists x \in P$? $\Longleftrightarrow$ $\exists x$: $Ax \leq b$?

We know: '$\exists x$: $Ax = b$?' $\longrightarrow$ Gauss Algorithm

$Ax \leq b$? $\longrightarrow$ VL: transform to Standard Form + find a Basis B: $x_B = B^{-1} \cdot b (\geq 0)$

How?

$\xrightarrow{OR}$ FME

• FME finds feas sol. to $Ax \leq b$ by projections of polytopes

Short digression: Projections (orthogonal)

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $k \leq n$

$\pi_k : \mathbb{R}^n \to \mathbb{R}^k$ maps x onto first k coordinates

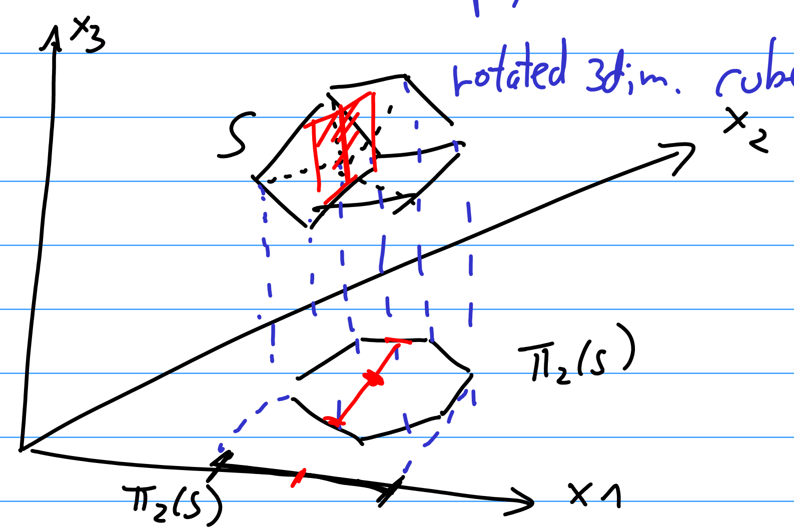
$\pi_k(x) = \pi_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$

extension to sets: $S \subset \mathbb{R}^n$, $\pi_k(S) := \{\pi_k(x) \mid x \in S\}$

equivalently: $\pi_k(S) = \{(x_1, \ldots, x_k) \mid \exists x_{k+1}, \ldots, x_n \text{ s.t. } (x_1, \ldots, x_n) \in S\}$

important note: S not empty $\Longleftrightarrow$ $\pi_k(S)$ not empty

Picture



rotated 3dim. cube, projected to $\pi_1(S)$ and $\pi_2(S)$

useful to decide wether $P \subset \mathbb{R}^n$ is empty or not

<u>idea:</u>
- eliminate x_n, project P to $\mathbb{R}^{n-1}$
- decide for "easier" polyhedron wether non-empty
  ($\to$ iterate projection)
- $Ax \le b$, $x \in \mathbb{R}^1 = \mathbb{R}$ is easy
  ($\to$ intersection of half-open intervals, e.g., $x_1 \le 7$, $x_1 \ge 4$)
- solution for P can be found by back-substitution after choosing a value for the last variable

$Ax \le b$, $\quad A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

w.l.o.g. all entries of the first column of A $(A_1) \in \{-1, 0, 1\}$

-1: $i \in M^-$, $\quad 0$: $i \in M$, $\quad 1$: $i \in M^+$

$$A = \begin{array}{c} A_1 \quad A_2 \;\text{——}\; A_n \\ \left(\begin{array}{c|ccc} 1 & * & & \\ 0 & & & \\ -1 & & & \\ 1 & & & \\ -1 & & & \\ 0 & & & \\ 1 & & & \\ 1 & & & \end{array}\right) \end{array} \begin{array}{l} \\ \leftarrow M^+ \\ \leftarrow M \\ \leftarrow M^- \\ \leftarrow M^+ \\ \vdots \\ \\ \\ \\ \end{array}$$

define $x' := (x_2, \dots, x_n)$
$A' := (A_2 | A_3 | \dots | A_n)$

$$P := \underbrace{Ax \le b}_{\text{canonical form}} \iff \begin{array}{rll} x_1 + a_i' x' \le b_i & & \forall i \in M^+ \\ -x_1 + a_i' x' \le b_i & & \forall i \in M^- \\ a_i' x' \le b_i & & \forall i \in M \end{array}$$

$$\iff \max_{i \in M^-} (a_i' x' - b_i) \le x_1 \le \min_{i \in M^+} (b_i - a_i' x')$$

$$a_i' x' \le b_i \qquad \forall i \in M$$

$\boxed{1}$

$\iff \quad a_i' x' - b_i \le b_j - a_j' x' \quad \forall\ i \in M^- \ \forall\ j \in M^+$

$a_i' x' \le b_i$

$\iff \quad \underbrace{(a_i' + a_j') x' \le (b_j + b_i) \quad \forall\ i \in M^-, j \in M^+ \qquad a_i' x' \le b_i \quad \forall\ i \in M}_{Q:\ \text{LP in canonical form}}$

P: n variables $\rightsquigarrow$ Q: $n-1$ variables

m constraints $\qquad |M| + |M^+| \cdot |M^-|$ constraints

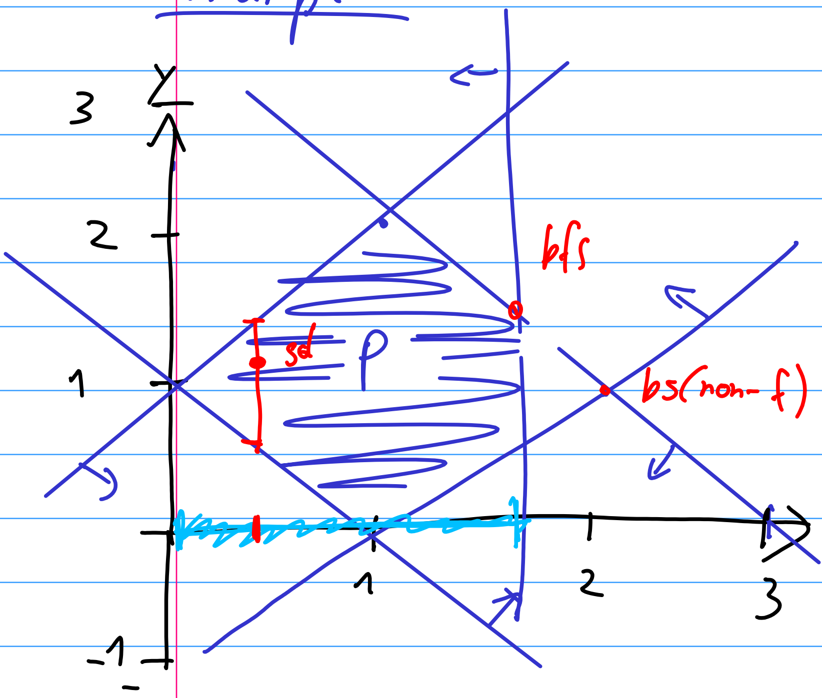
$(Ax \le b)$ $\qquad \left(\begin{pmatrix} a_i' + a_j' \\ a_i' \end{pmatrix} x' \le \begin{pmatrix} b_j + b_i \\ b_i \end{pmatrix} \begin{matrix} i \in M^-, j \in M^+ \\ i \in M \end{matrix} \right)$

- use steps to eliminate $n-1$ var., choose value for the last
- use bounds of the form $\boxed{1}$ to choose other variables

$\rightsquigarrow$ polynomial running time?:

- no, exponential
- in every projection step, the number of constraints might double $\rightsquigarrow 2^{(n-1)}$ times bigger

example 1:



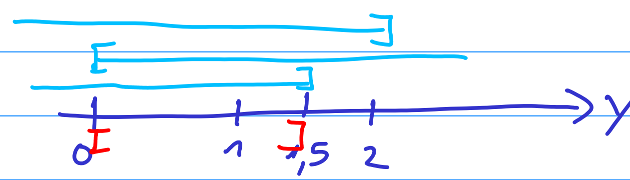
$x + y \geq 1$
$x + y \leq 3$ (x)
$x - y \geq -1$
$x - y \leq 1$
$y \leq 1.5$
$\Updownarrow$

(1)	$-x - y \leq -1$	M^-
(2)	$x + y \leq 3$	M^+
(3)	$-x + y \leq 1$	M^-
(4)	$x - y \leq 1$	M^+
(5)	$y \leq 1{,}5$	M

$M^- = \{(1), (3)\}$, $M^+ = \{(2), (4)\}$
$M = \{(5)\}$

First projection: $(a'_i + a'_j)\, x' \leq (b_i + b_j)$, $i \in M^+, j \in M^-$
$a'_i\, x' \leq b_i$ $\quad i \in M$

(1),(2)	$\cancel{0y \leq 2}$	$0 \leq 2$	
(1)(4)	$-2y \leq 0$		$y \geq 0$
(3),(2)	$2y \leq 4$		$y \leq 2$
(3)(4)	$\cancel{0y \leq 2}$	$0 \leq 2$	$y \leq 1.5$
(5)	$y \leq 1{,}5$		



$\longrightarrow \quad 0 \leq y \leq 1.5$

choose $y = 0.5$ $\overset{(*)}{\Longrightarrow}$

$-x \leq -0.5$		$x \geq 0.5$
$x \leq 2.5$	$\leadsto$	$x \leq 2.5$
$-x \leq 0.5$		$x \geq -0.5$
$x \leq 1.5$		$x \leq 1.5$

$x \in [0.5; 1.5]$

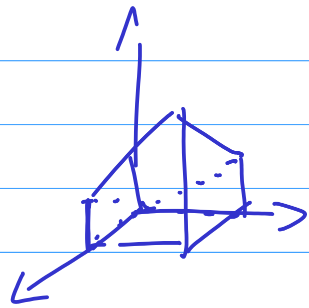
$\Rightarrow$ choose $x = 1.1$

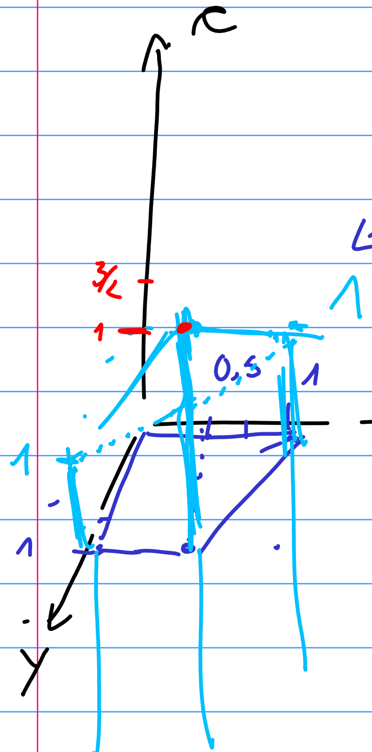
$\Rightarrow \quad (1.1; 0.5) \in P$

Remark: FME shows: $\boxed{\text{projections of polyhedra are polyhedra}}$

- Solving LPs via FME:

<u>example 2</u>





↙ unbounded along $-c$

$$\max\ x+y \quad \text{s.t.}$$
$$x \geq 0$$
$$y \geq 0$$
$$y \leq 1$$
$$x + \tfrac{1}{2}y \leq 1$$

= for $(1,0)$, $(0.5,1)$

add constraint for obj. value
$$c-(x+y) \leq 0$$
($c=0 \rightarrow x,y \geq 0$, other const.)

$$\begin{array}{rrrrl}
-x & & & \leq 0 & (M^-)\ 1 \\
 & -y & & \leq 0 & (M) \\
 & y & & \leq 1 & (M) \\
x & +\tfrac{1}{2}y & & \leq 1 & (M^+)\ 3 \\
-x & -y & +c & \leq 0 & (M^-)\ 2
\end{array} \quad \rightsquigarrow \quad
\begin{array}{rrll}
 & & & \text{redundant} \\
\cancel{\tfrac{1}{2}y} & & \cancel{\leq 1} & \cancel{(1,3)} \\
-\tfrac{1}{2}y & +c & \leq 1 & (2,3) \\
-y & & \leq 0 & \\
y & & \leq 1 &
\end{array}$$

$$\rightsquigarrow \begin{array}{rrll}
-y & +2c & \leq 2 & - \\
-y & & \leq 0 & : \\
y & & \leq 1 & +
\end{array} \quad \rightsquigarrow \quad
\begin{array}{l}
2c \leq 3 \Leftrightarrow c \leq 3/2 \\
\cancel{0 \leq 1} \ \text{red.}
\end{array}$$

$\Rightarrow$ choose $c = 3/2$

$\rightsquigarrow$ $-y \leq -1 \; (y \geq 1)$ $\qquad y = 1$
$y \geq 0$
$y \leq 1$

$\rightsquigarrow$ $x \geq 0$
$x \leq 1/2$ $\qquad x = 1/2$
$x \geq 1/2$

$(0.5, 1, 1.5) \in P$, opt. sol. for LP: $(0.5, 1)$
with $x + y = 0.5 + 1 = 1.5 = c$

1. $\min c^T x$ $\quad Ax \leq b$ $\quad x \geq 0$ $\quad \rightsquigarrow \quad \tilde{A} x \leq \tilde{b}$, $\quad \lambda - c^T x \geq 0$
2. eliminate all variables except for λ
   system: $A \lambda \leq b$
3. choose smallest possible value for λ
4. extend λ to a valid solution $(x_1, \dots, x_n, \lambda)$
   now: $\lambda = c^T x$