Ex. Session 6 20.11.13

The network simplex

(Friday 01os, Ex. Sheets → MA 503)

Network Simplex (solving Min. Cost Flow Problems)

MinCost Flow Problem:

Instance: G, c, b $G = (V, E)$ directed graph with $|V| = n$ and $|E| = m$

$c : E \to \mathbb{R}^+$ cost for edges

$b : V \to \mathbb{R}$ balances for all vertices

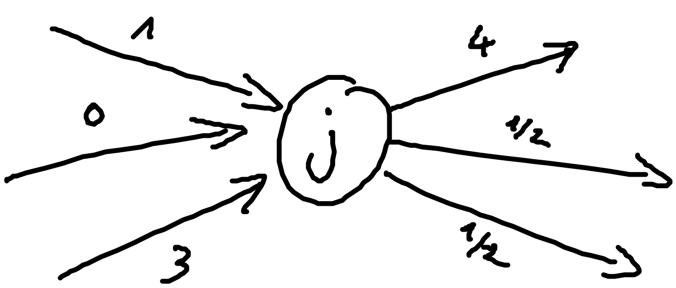
$b(v) > 0$ then v has a supply

< 0 v has a demand

($u: E \to \mathbb{R}^+$ capacities) for today $u \equiv \infty$

Solution: Flow $f: E \to \mathbb{R}_0^+$ s.t. $\underbrace{\sum_{e=(j,i)}}_{\text{Outgoing}} f(e) - \underbrace{\sum_{e=(i,j)}}_{\text{Ingoing}} f(e) = b(j)$ $\forall j \in V$

$b(j) = 1$

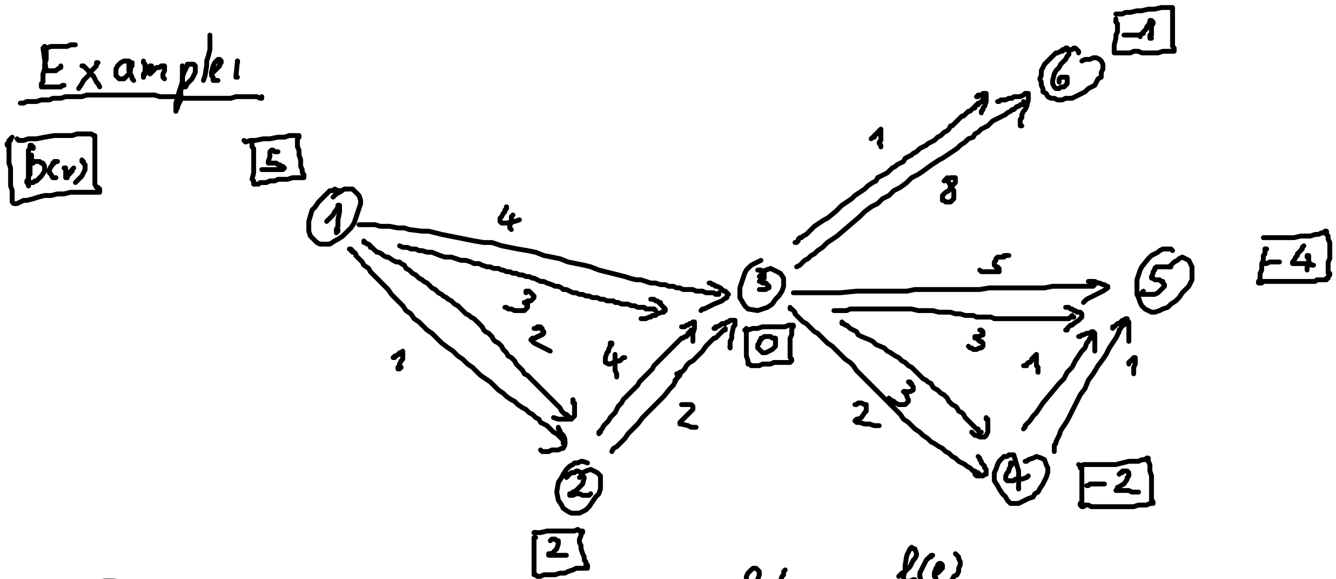


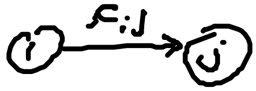
$5 - 4 = 1 = b(j)$

$0 \leq f(e) \ (\leq u(e))$

costs: $\sum_{e \in E} c(e) \cdot f(e)$

Example:





flow: $\xrightarrow{f(e)}$

LP formulation:

$$\min \; c^T f$$

$$A \cdot f = b$$

$$(u \geq)\; f \geq 0$$

$$A = \begin{array}{c|c} & e_1 \; \dots, e_k = (i,j), \dots \; e_m \\ \hline \begin{array}{c} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_j \\ \vdots \\ v_n \end{array} & \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ \dots & 1 & \dots \\ & 0 & \\ & \vdots & \\ & 0 & \\ \dots & -1 & \dots \\ & 0 & \end{pmatrix} \end{array}$$

A is a node-edge incidence matrix

→ sum of all rows is $(0, \dots, 0)$

delete row n

remark: flow conservation in node v_n is a consequence of the flow conservation in all other nodes

new system: $\bar{A} f = \bar{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}$, $\bar{A}$ has $n-1$ lin. independent rows

Assume:
- $\sum b(i) = 0$ (no solution otherwise)
- Graph G is connected (several subproblems)

Tree and basic feas. solutions

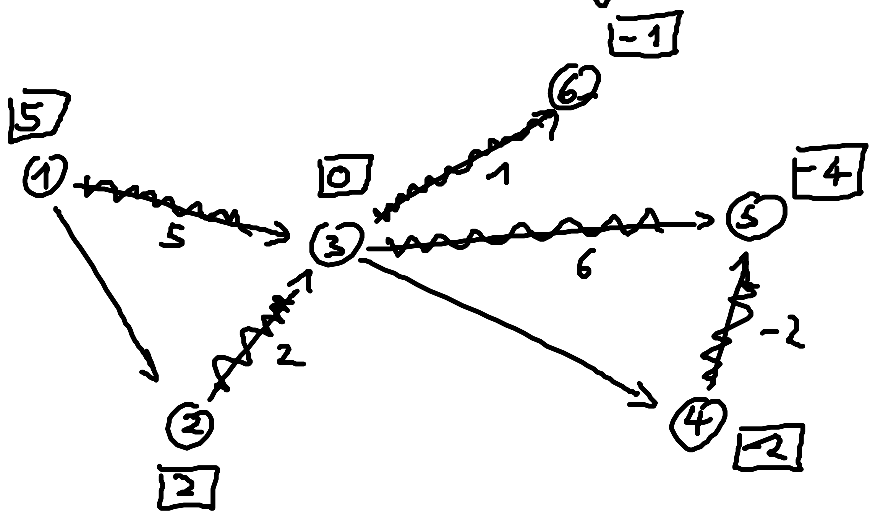
Def: f is tree sol. Iff it can be constructed by this procedure

(a) pick $T \subset E$, $|T| = n-1$, T is a tree

(b) $f_{ij} = 0 \quad \forall (i,j) \notin T$

(c) solve $\bar{A} f = \bar{b}$ to get f_{ij} for $(i,j) \in T$

Example:



procedure: • select root (1)
• calculate flow in preorder sequence of the nodes (T_L, T_R, root)

• solution is uniquely determined by the tree, not ness. a feas. solution ($f(e) < 0$)

Thm: $T \subset E$ is a set of $n-1$ edges that form a tree
$\Rightarrow \bar{A} f = \bar{b}$ and $f_{ij} = 0 \quad \forall (i,j) \notin T$ has a unique sol.

Thm: a flow is a bfs Iff it is a feas. tree sol.

Proof: „$\Leftarrow$" • $f_{ij} = 0$ for all non-tree edges
• $n-1$ edges in tree, with unique sol
→ size correct, edges independent structure

„$\Rightarrow$" suppose f is a bfs

$S := \{(i,j) \in E \mid f_{ij} \neq 0\}$

- case 1: no cycle in S, select $T \supseteq S$ s.t. $|T| = n-1$, T is a tree

  $\rightarrow$ there is a unique sol.

- case 2: S contains cycle:

$$f := f' + f_c$$

$$\bar{A} f = \bar{A} f' + \underbrace{A f_c}_{=0} = b$$

another sol. f', at least 1 edge less then before with a positive flow

$\Rightarrow$ f was no bfs $\lightning$

$\square$

<u>so far:</u>

- correspondence bfs $\Leftrightarrow$ feas. tree sol.
- given basis B, $f = B^{-1} b$ can be computed easily
  - $\rightarrow$ no need for B^{-1}
  - $\rightarrow$ no numerical problems

<u>change of basis:</u>

- choose an edge, curr. not in tree T
- edge $e = (i,j)$ enters basis, another leaves T

$T + \{e\}$ has a cycle C

- $f_e = 0$, if $e' \in C$ $f_{e'} = 0$
  $\rightarrow$ delete e' from T

otherwise: send flow along cycle, C is oriented in the dir. of e

$$\bar{f}_{kl} = \begin{cases} f_{kl} + \delta & (k,l) \in F \\ f_{kl} - \delta & (k,l) \in B \\ f_{kl} & (k,l) \notin C \end{cases}$$

- B are backward edges in cycle C
- F are forward edges in cycle C

$$\underbrace{\delta_0 = \min_{(k,l)\in B}}_{\text{needed for } f\geq 0} f_{kl}, \quad \text{(max. flow change)}$$

<u>cost change:</u> changes by $\underbrace{\delta_0}_{\text{flow amount}} \underbrace{\left(\sum_{(k,l)\in F} c_{kl} - \sum_{(k,l)\in B} c_{kl} \right)}_{\bar{c}_{ij} \sim \text{ cost for cycle}}$

another formula for $\bar{c}$:

reminder: $\bar{c} = c - \pi' A$ $\qquad \pi_i \quad 1 \leq i \leq n-1$ (for each node)

$\pi' = c_B' \cdot B^{-1}$ $\qquad$ (i,j) is the k-th edge

$\Rightarrow$ k-th entry $\pi' A$

$$\bar{c}_{ij} = \begin{cases} c_{ij} - (\pi_i - \pi_j) & i,j \neq n \\ c_{ij} - \pi_j & j = n \\ c_{ij} + \pi_j & i = n \end{cases}$$

$\rightsquigarrow$ solving $\pi_n = 0$, $\pi_i - \pi_j = c_{ij}$ gives red. cost

<u>Algorithm:</u>

① start with bfs f associated with tree T

② compute π by proceeding from root towards leaves

③ compute $\bar{c}_{ij}$

- $\bar{c} \geq 0 \Rightarrow$ opt. sol. found
- $\bar{c}_{ij} < 0$ for (i,j), (i,j) enters T,
  - sends flow along cycle
  - another var. leaves basis

→ If no backward edges ⇒ unbounded

④ $\delta_0 = \min_{(k,\ell)\in B} f_{k\ell}$, push δ_0 along cycle, remove (k,ℓ) from T

remarks: introduce new node $n+1$

edges: $(i, n+1)$ $\forall i \in V: b(i) \leq 0$

$(n+1, i)$ $\forall i \in V: b(i) > 0$

flow: $f_{(i,n+1)} = |b(i)|$

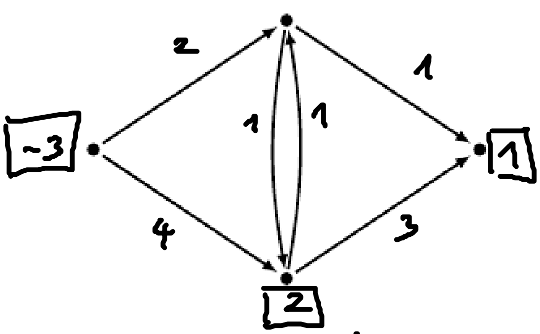
$f_{(n+1,i)} = |b(i)|$

$f = 0$ for the rest

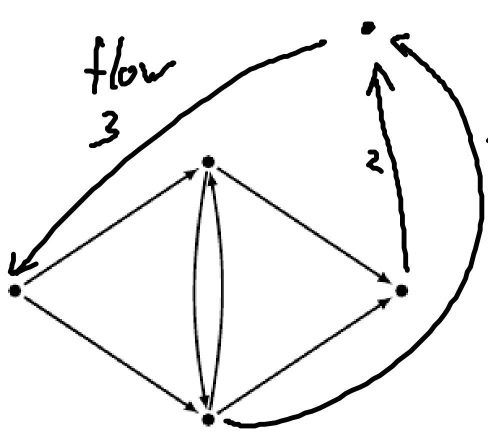
cost on the edges: $c = \sum_{e \in E} c(e) + 1$

Examples:

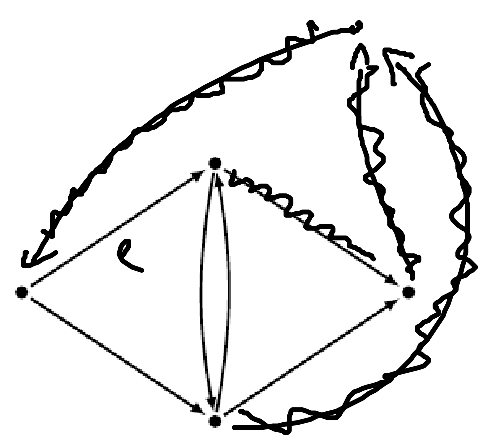
instance



flow



tree



$\bar{c}_e = 2 + M + M + M$

$\bar{c}_e > 0 \quad \forall e \in E$

$\rightsquigarrow$ no valid sol., can not satisfy
inflow of 3 at vertex 1 w.o.
artifical edges