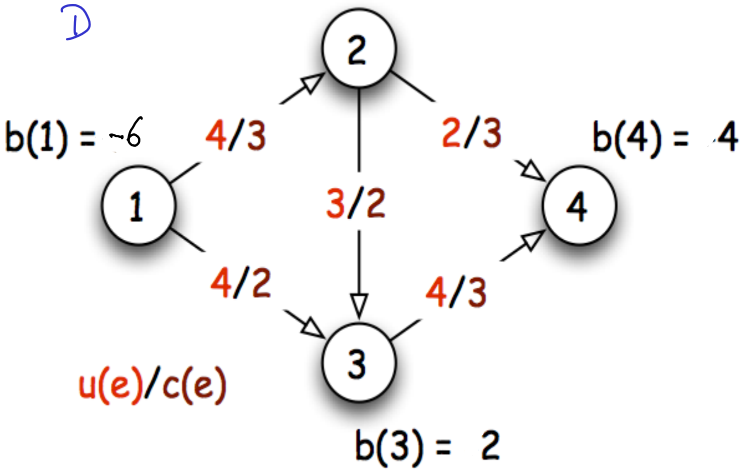
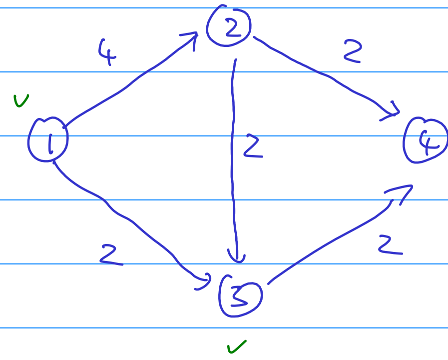
D

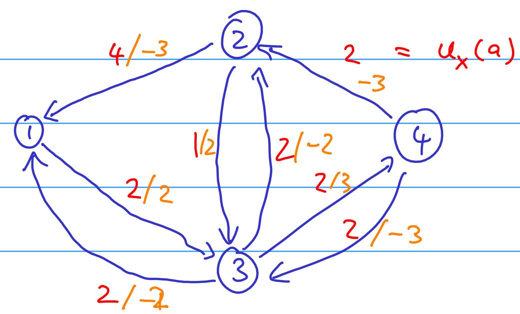


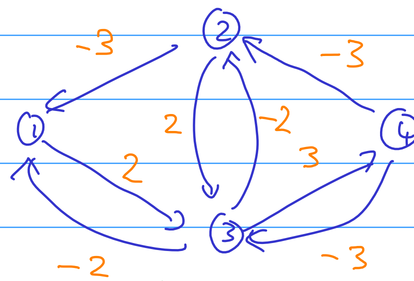
x

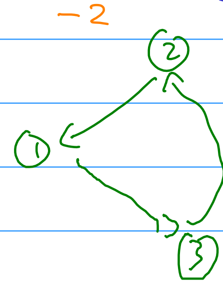


$c(x) = 32$

D_x



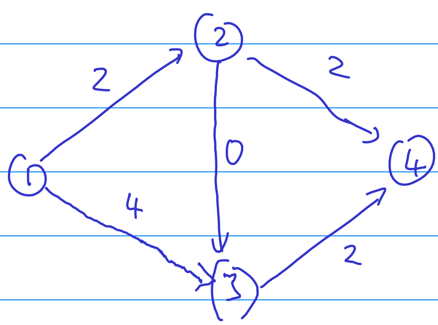




negative Zykel

after augmentation :

x'



$cost(x') = 26$

Negative-Cycle Canceling Algorithm

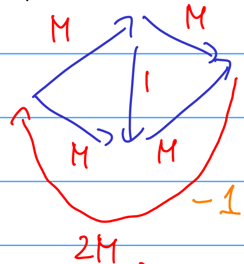
Negative-cycle canceling algorithm

- i compute a feasible b-transshipment x or determine that none exists;
- ii while there is a negative-cost dicircuit C in D_x
- iii set $x := x + \delta \cdot \chi^C$ with $\delta := \min\{u_x(a) \mid a \in C\}$;

Remarks:

- The negative-cost dicircuit C in step (ii) can be found in $O(nm)$ time by the Ford-Bellman Algorithm.
- However, the number of iterations is only pseudo-polynomial in the input size.
- If arc capacities and b-values are integral, the algorithm returns an integral min-cost b-transshipment.

Bad example:

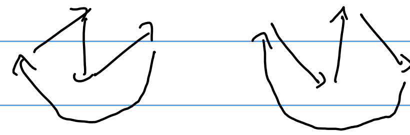


cost 0 on blue arcs

−1 cost

2M — capacities

$\Rightarrow$ 2M iterations in worst case by chosing



alternatingly

Better choice of cycles:
choose dicycle with minimum mean cost

Minimum-Mean-Cycle Canceling Algorithm

The mean cost of a dicircuit C in D_x is

$$\frac{\sum_{a \in C} c(a)}{|C|}$$

Theorem 7.5.
Choosing a minimum mean-cost dicircuit in step (ii), the number of iterations is in $O(n \cdot m^2 \cdot \log n)$.

Proof: . . . see below $\square$

Theorem 7.6.
A minimum mean-cost dicircuit can be found in $O(n \cdot m)$ time.

Proof: Exercise! $\square$

Running Time of Minimum-Mean-Cycle Canceling

Corollary 7.7.

A min-cost b-transshipment can be found in $O(n^2 \cdot m^3 \cdot \log n)$ time.

Remark:

- Goldberg and Tarjan showed that the running time of the minimum-mean cycle canceling algorithm can be improved to $O(n \cdot m^2 \cdot \log^2 n)$.

Proof of Thm 7.5

Fact: D digraph with $|\delta^-(v)| = |\delta^+(v)| \quad \forall v$

indegree   outdegree

$\Rightarrow$ D can be decomposed into arc-disjoint elementary dicycles

Let $x_1 \xrightarrow{C_1} x_2 \xrightarrow{C_2} \dots$ the sequence of flows x_i and dicircuits C_i in D_{x_i} in the algorithm

Let $A_i := A_{x_i} \Rightarrow C_i$ is mmc in A_i

minimum mean cycle

let μ_i be its mmc - cost

Lemma 1 (note the resemblance to Edmonds-Karp)

(a) $\mu_k \leq \mu_{k+1}$ (weak monotonicity)

(b) $\mu_k \le \left(\frac{n}{n-1}\right) \mu_r$ for all $k < r$ such that $C_k \cup C_r$ contains a pair a, a^{-1}

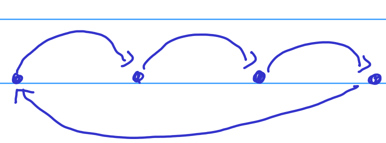
(strong monotonicity)

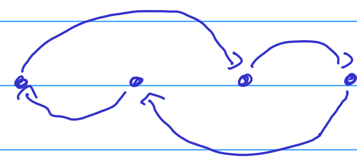
Proof Lemma 1:

a) consider x_k and x_{k+1}

let $H := C_k + C_{k+1} - \{\text{pairs } a, a^{-1}\}$ (considered as vector sum)

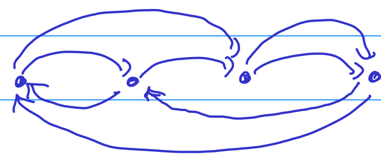
C_k



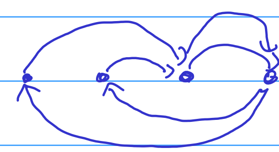


C_{k+1}

$C_k + C_{k+1}$



H



Remark: H fulfils assumptions in Fact!

$\Rightarrow$ H is union of arc disjoint elementary dicycles

(1) $\boxed{(1)\ H \subseteq A_k}$ (H considered as set of arcs)

$$A_k \xrightarrow[C_k]{} A_{k+1} \xrightarrow[C_{k+1}]{} A_{k+2}$$

assume $a \in H \setminus A_k \Rightarrow a \in C_{k+1} \setminus A_k$

$\Rightarrow$ a is reverse arc of some arc in C_k

$\Rightarrow$ $a, a^{-1} \in H$, contradiction to deletion of such pairs

(2) $\boxed{c(H) \ge \mu_k |H|}$

Fact $\Rightarrow$ H is arc-disjoint union of elementary dicycles in A_k

say $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_s$

$\Rightarrow c(\bar{C}_i) \ge \mu_k |\bar{C}_i| \qquad i = 1, 2, \dots, s$

C_k is mmc

$$\Rightarrow c(H) = c(\bar{C}_1) + \dots + c(\bar{C}_s)$$
$$\geq \mu_k (|\bar{C}_1| + \dots + |\bar{C}_s|) = \mu_k |H|$$

(3) $\boxed{|H| \leq |C_k| + |C_{k+1}|}$

$\uparrow$ pairs a, a^{-1} were deleted

(4) $\boxed{c(H) = \mu_k |C_k| + \mu_{k+1} |C_{k+1}|}$

since $c(H) = c(C_k) + c(C_{k+1})$

$\uparrow$ cost of pairs a, a^{-1} are 0

$$= \mu_k |C_k| + \mu_{k+1} |C_{k+1}|$$

Combining (2) - (4) gives

$$\mu_k (|C_k| + |C_{k+1}|) \underset{\uparrow (3)}{\leq} \mu_k |H| \underset{\uparrow (2)}{\leq} c(H)$$
$$\overset{(4)}{=} \mu_k |C_k| + \mu_{k+1} |C_{k+1}|$$

$$\Rightarrow \mu_k |C_{k+1}| \leq \mu_{k+1} |C_{k+1}| \Rightarrow \mu_k \leq \mu_{k+1}$$

b) may assume because of a) that $C_k + C_i$ contains no pair a, a^{-1} $\forall$ $k < i < r$

Consider again $H := C_k + C_r - \{\text{pairs } a, a^{-1}\}$

(1) $H \subseteq A_k$

if $a \in C_r \setminus A_k$ $\Rightarrow$ a is reverse arc of some arc in $C_{k+1}, C_{k+2}, \dots, C_r$

$\Rightarrow$ (assumption on k, r) a is reverse arc of some arc in C_r

$\Rightarrow$ contradiction to deletion of pairs a, a^{-1}

As in a) we obtain

(2) $c(H) \geq \mu_k |H|$

(4) $c(H) = \mu_k |C_k| + \mu_{k+1} |C_{k+1}|$

Instead of (3), we get

(3') $|H| \leq \frac{n-1}{n} \left(|C_k| + |C_{k+1}| \right)$

at least 2 arcs are deleted from $C_k + C_r$

$C_k + C_r$ contains at most $2n$ arcs

$\Rightarrow |H| \leq \frac{2n-2}{2n} \left(|C_k| + |C_r| \right)$

As in a) we obtain

$$\mu_k \left(\frac{n-1}{n} \right) \left(|C_k| + |C_r| \right) \leq \dots \leq \mu_k |C_k| + \mu_r |C_r| \underset{(a)}{\leq} \mu_r \left(|C_k| + |C_r| \right)$$

$\Rightarrow \mu_k \leq \frac{n}{n-1} \mu_r \quad \Rightarrow (6)$

$\Updownarrow$

$|\mu_r| \leq \frac{n-1}{n} |\mu_k|$ absolute value decreases by factor $\left(1 - \frac{1}{n}\right)$

Lemma 2

$|\mu_k|$ decreases in $m \cdot n$ iterations by at least $1/2$

Proof: consider $C_k, C_{k+1}, \dots, C_{k+m}$ of $m+1$ dicycles in the algorithm

every of these C_i gets one arc saturated

$\Rightarrow$ reverse arc is in A_{i+1}

$m+1$ dicycles, only m arcs in D

$\Rightarrow$ 2 of these dicycles must contain a pair a, a^{-1}

say $C_i, C_j \quad i < j$

$\Rightarrow$ (Lemma 1) $\mu_k \leq \mu_i \leq \frac{n}{n-1} \mu_j \leq \frac{n}{n-1} \mu_{k+m}$

$\Rightarrow \quad |\mu_k| \geq \frac{n}{n-1} |\mu_{k+m}| \quad \Rightarrow \quad |\mu_{k+m}| \leq \frac{n-1}{n} |\mu_k|$

$\Rightarrow$ decrease by a factor $\frac{n-1}{n}$ every m iterations

$\Rightarrow$ decrease by a factor $\left(\frac{n-1}{n}\right)^n$ every $n \cdot m$ iterations

$\left(1-\frac{1}{n}\right)^n \nearrow e^{-1} = 0{,}37 < \frac{1}{2}$

Lemma 3: After $m \cdot n (\lceil \log n \rceil + 1)$ iterations, at least one arc a is fixed, i.e. the flow on it remains unchanged

Proof: consider x at some iteration and x' $m \cdot n (\lceil \log n \rceil + 1)$ iterations later

define cost $c'(a) := c(a) - \mu(x')$

$\uparrow$ mmc cost in $D_{x'}$

$\Rightarrow$ no negative dicycles in $(D_{x'}, c')$

$\uparrow$ if not, then $\exists$ C wit $c'(C) < 0$

$\Rightarrow c'(C) = c(C) - |C| \cdot \mu(x') < 0$

$\Rightarrow \frac{c(C)}{|C|} < \mu(x')$

$\Rightarrow \exists$ feasible potential in $(D_{x'}, c')$

$\Rightarrow 0 \leq c'_\pi(a) = c_\pi(a) - \mu(x')$

(1) $\Rightarrow c_\pi(a) \geq \mu(x') \quad \forall a \in D_{x'}$

Let C be the chosen mmc in D_x

$\Rightarrow \mu(x) \underset{\text{lemma 3}}{\leq} \underbrace{2^{\lceil \log n \rceil + 1}}_{\leq 2n} \mu(x') \leq 2n \, \mu(x')$