Chapter 4:
Duality Theory

(cp. Bertsimas & Tsitsiklis, Chapter 4)

Motivation

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, consider the linear program

$$\min \ c^T \cdot x \qquad \text{s.t.} \quad A \cdot x \geq b, \ x \geq 0$$

Question: How to derive lower bounds on the optimal solution value?

Idea: For $p \in \mathbb{R}^m$ with $p \geq 0$: $\quad A \cdot x \geq b \quad \Longrightarrow \quad (p^T \cdot A) \cdot x \geq p^T \cdot b$

Thus, if $c^T \geq p^T \cdot A$, then

$$c^T \cdot x \geq (p^T \cdot A) \cdot x \geq p^T \cdot b \qquad \text{for all feasible solutions } x.$$

Find the best (largest) lower bound in this way:

$$\begin{array}{ll} \max & p^T \cdot b \\ \text{s.t.} & p^T \cdot A \leq c^T \\ & p \geq 0 \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ll} \max & b^T \cdot p \\ \text{s.t.} & A^T \cdot p \leq c \\ & p \geq 0 \end{array}$$

This LP is the dual linear program of our initial LP.

Primal and Dual Variables and Constraints

primal LP (minimize)		dual LP (maximize)	
	$\geq b_i$	≥ 0	
constraints	$\leq b_i$	≤ 0	variables
	$= b_i$	free	
	≥ 0	$\leq c_i$	
variables	≤ 0	$\geq c_i$	constraints
	free	$= c_i$	

— used in the motivation

Basic Properties of the Dual Linear Program

Theorem 4.1.
The dual of the dual LP is the primal LP.

Proof:...

Theorem 4.2.
Let Π_1 and Π_2 be two LPs where Π_2 has been obtained from Π_1 by (several) transformations of the following type:

i. replace a free variable by the difference of two non-negative variables;
ii. introduce a slack variable in order to replace an inequality constraint by an equation;
iii. if some row of a feasible equality system is a linear combination of the other rows, eliminate this row.

Then the dual of Π_1 is equivalent to the dual of Π_2.

Proof:...

Proof of Thm 4.1: Assume primal is of general form with = and ≥ constraints

- Write the dual in primal form:

$$
\begin{array}{lrcll}
\min\ \pi^T(-b) & & \text{such that} & \\
(-A_j^T)\pi & \geq & -c_i & j \in N \\
(-A_j^T)\pi & = & -c_i & j \in \overline{N} \\
\pi_i & \geq & 0 & j \in \overline{M} \\
\pi_i & & \text{unconstrained} & j \in M
\end{array}
$$

- The transformation rules yield the following dual LP

$$
\begin{array}{lrcll}
\max\ x^T(-c) & & \text{such that} & \\
x_j & \geq & 0 & j \in N \\
x_j & & \text{unconstrained} & j \in \overline{N} \\
-a_i^T x & \leq & -b_i & i \in \overline{M} \\
-a_i^T x & = & -b_i & i \in M
\end{array}
$$

which is the primal LP ❑

$\pi \overset{\wedge}{=} p$

primal

$\min\ c^T x$

$a_i^T x = b_i \qquad i \in \overline{M}$

$a_i^T x \geq b_i \qquad i \in M$

$x_j \geq 0 \qquad j \in N$

x_j free $\qquad j \in \overline{N}$

Proof of Thm 4.2

(i) splitting variables

$\min\ c^T x$
s.t. $Ax \geq b$
x free

$\xrightarrow{\text{transform}}$

$x = x^+ - x^-$
$x^+, x^- \geq 0$

$\min\ c^T x^+ - c^T x^-$
s.t. $Ax^+ - Ax^- \geq b$
$x^+, x^- \geq 0$

$\Updownarrow$

$$\min \begin{pmatrix} c \\ -c \end{pmatrix}^T \begin{pmatrix} x^+ \\ x^- \end{pmatrix}$$

$$\text{st}\ \ (A \mid -A)\begin{pmatrix} x^+ \\ x^- \end{pmatrix} \geq b$$

$$\begin{pmatrix} x^+ \\ x^- \end{pmatrix} \geq 0$$

dualize $\downarrow$

dualize $\downarrow$

$$\max \; p^T b \quad \text{s.t.} \; \boxed{p^T A = c^T} \quad p \geq 0 \quad \overset{\text{equivalent}}{\longleftrightarrow} \quad \max \; p^T b \quad \text{s.t.} \; \boxed{p^T (A \mid -A) \geq \begin{pmatrix} c \\ -c \end{pmatrix}} \quad p \geq 0$$

(ii) adding slack variables

$$\min \; c^T x \quad \text{s.t.} \; Ax \geq b, \; x \text{ free} \quad \xrightarrow[\text{add slack variables}]{\text{transform}} \quad \min \; c^T x + 0^T y \quad \text{s.t.} \; Ax - Iy = b, \; y \geq 0$$

dualize $\downarrow$ ... dualize $\downarrow$

$$\max \; p^T b \quad \text{s.t.} \; p^T A = c^T, \; \boxed{p \geq 0} \quad \overset{\text{equiv.}}{\longleftrightarrow} \quad \max \; p^T b \quad p^T A = c^T, \; \boxed{p^T(-I) \leq 0^T}, \; p \text{ free}$$

(iii) eliminating redundant rows

$$\min \; c^T x \quad \text{s.t.} \; Ax = b, \; x \geq 0$$

assume row m is a linear combination of rows $1, \ldots, m-1$

i.e.
$$\boxed{a_m^T = \sum_{i=1}^{m-1} y_i a_i^T \qquad b_m = \sum_{i=1}^{m-1} y_i b_i} \qquad (*)$$

$\downarrow$ dualize

$$\max \sum_{i=1}^{m} p_i \cdot b_i \overset{(*)}{=} \sum_{i=1}^{m-1} (p_i + \gamma_i p_m)\, b_i$$

$$\text{s.t.} \sum_{i=1}^{m} p_i\, a_i^T \leq c^T \overset{(*)}{\longrightarrow} \sum_{i=1}^{m-1} \underbrace{(p_i + \gamma_i p_m)}_{=: q_i}\, a_i^T \leq c^T$$

$$\Rightarrow \max \sum_{i=1}^{m-1} q_i\, b_i \qquad \text{s.t.} \sum_{i=1}^{m-1} q_i\, a_i^T \leq c^T$$

dual of original problem with last row deleted

Weak Duality Theorem

Theorem 4.3.

If x is a feasible solution to the primal LP (minimization problem) and p a feasible solution to the dual LP (maximization problem), then

$$c^T \cdot x \geq p^T \cdot b \enspace .$$

Proof:. . .

Corollary 4.4.

Consider a primal-dual pair of linear programs as above.

- a If the primal LP is unbounded (i. e., optimal cost $= -\infty$), then the dual LP is infeasible.
- b If the dual LP is unbounded (i. e., optimal cost $= \infty$), then the primal LP is infeasible.
- c If x and p are feasible solutions to the primal and dual LP, resp., and if $c^T \cdot x = p^T \cdot b$, then x and p are optimal solutions.

clear

Proof

Theorem 4.2 $\Rightarrow$ may assume that primal (P) and dual (D) are of the form

(P) $\min c^T x$
s.t. $Ax \geq b$
$x \geq 0$

(D) $\max \pi^T b$
s.t. $p^T A \leq c^T$
$p \geq 0$

Let x, p be feasible for (P), (D). Then

$$c^T x \underset{x \geq 0}{\geq} (p^T A) x = p^T (Ax) \underset{Ax = b}{=} p^T b \qquad \square$$

Strong Duality Theorem

Theorem 4.5.
If an LP has an optimal solution, so does its dual and the optimal costs are equal.

Proof:. . .

dual (dual) = primal $\Rightarrow$ may assume that the primal (P) has an optimal solution

Thm. 4.2 $\Rightarrow$ may assume that (P) is in standard form

(P) $\min c^T x$
s.t. $Ax = b$
$x \geq 0$

optimum attained on some bfs

$\Rightarrow$ may assume that optimal solution y is a bfs with basis B

y optimal $\Rightarrow$ reduced cost $\boxed{\bar{c}^T = c^T - \underbrace{c_B^T B^{-1}}_{=: q^T} A \geq 0}$ (*)

$\Downarrow$

$q^T A \leq c^T$

$\Rightarrow$ q is feasible in (D)

$$(D) \quad \begin{array}{ll} \max & p^T b \\ \text{s.t.} & p^T A \leq c^T \\ & p \text{ free} \end{array}$$

Then $q^T b = (c_B^T B^{-1}) b = c_B^T (B^{-1} b) = c_B^T y_B = c^T y$

values of basic variables in y

non basic $y_j = 0$

$\Rightarrow$ $q^T b = c^T y$

$\Rightarrow$ q is optimal in (D) $\square$

Different Possibilities for Primal and Dual LP

primal \ dual	finite optimum	unbounded	infeasible
finite optimum	strong duality possible	impossible	impossible
unbounded	impossible	impossible	possible ?
infeasible	impossible	possible ?	possible ?

partly exercises

Example of infeasible primal and dual LP:

min $\quad x_1 + 2\,x_2$ $\qquad\qquad$ max $\quad p_1 + 3\,p_2$

$$\begin{array}{llcl} \text{s.t.} & x_1 + x_2 = 1 & \quad \text{s.t.} & p_1 + 2\,p_2 = 1 \\ & 2\,x_1 + 2\,x_2 = 3 & & p_1 + 2\,p_2 = 2 \end{array}$$

Complementary Slackness

Consider the following pair of primal and dual LPs:

$$\begin{array}{lll} \min & c^T \cdot x & \qquad \max & p^T \cdot b \\ \text{s.t.} & A \cdot x \geq b & \qquad \text{s.t.} & p^T \cdot A = c^T \\ & & & p \geq 0 \end{array}$$

If x and p are feasible solutions, then $c^T \cdot x = p^T \cdot A \cdot x \geq p^T \cdot b$.Thus,

$$c^T \cdot x = p^T \cdot b \quad \Longleftrightarrow \quad \text{for all } i\text{: } p_i = 0 \text{ if } {a_i}^T \cdot x > b_i.$$

Theorem 4.6.

Consider an arbitrary pair of primal and dual LPs. Let x and p be feasible solutions to the primal and dual LP, respectively. Then x and p are both optimal if and only if

(1) $u_i := p_i \, (\overbrace{{a_i}^T \cdot x - b_i}^{\text{primal slack}}) = 0$ for all i, (dual variable)·(primal slack)

(2) $v_j := (\underbrace{c_j - p^T \cdot A_j}_{\text{dual slack}}) \, x_j = 0$ for all j. (dual slack)·(primal var.)

Proof:. . . with dual variables π_i instead of p_i

- $u_i \geq 0$
  - since
  - $a_i^T x - b_i = 0 \Rightarrow u_i = 0$
  - $a_i^T x - b_i \geq 0 \Rightarrow \pi_i \geq 0 \Rightarrow u_i \geq 0$
- $v_j \geq 0$
  - since
  - x_j unconstrained $\Rightarrow \pi^T A_j = c_j \Rightarrow v_j = 0$
  - $x_j \geq 0 \Rightarrow \pi^T A_j \leq c_j \Rightarrow v_j \geq 0$

Set $u := \sum_i u_i$, $v := \sum_j v_j$ => $u, v \geq 0$. Then

$u = 0$ <=> (1) holds

$v = 0$ <=> (2) holds

Then

$$
\begin{aligned}
u + v &= \sum_i \pi_i \cdot (a_i^T x - b_i) + \sum_j (c_j - \pi^T A_j) \cdot x_j \\
&= - \sum_i \pi_i b_i + \sum_j c_j x_j + \sum_i \pi_i a_i^T x - \sum_j \pi^T A_j x_j \\
&= - \pi^T b + c^T x + (\pi^T A) x - \pi^T (Ax) \\
&= - \pi^T b + c^T x
\end{aligned}
$$

Hence: $u + v = - \pi^T b + c^T x$

Suppose (1) and (2) hold => $u + v = 0$ => $c^T x = \pi^T b$

Weak Duality Theorem => x, π are optimal

Suppose that x and π are optimal

Strong Duality Theorem => $c^T x = \pi^T b$ => $u + v = 0$ => (4.8) and (4.9) ❑