An example of our concepts so far

A simple linear programming problem:

objective $\rightarrow$

$$
\begin{array}{lrcrcrcr}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & \leq & 20 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

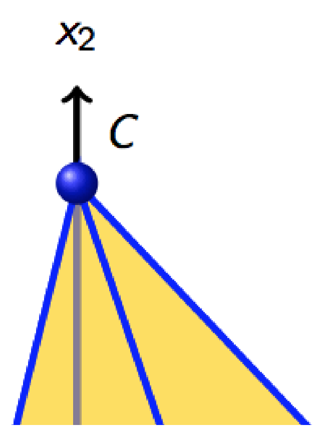
$\leftarrow$ $=$ defines hyperplane

$\leq$ defines halfspace

polyhedron $P = \bigcap$ of halfspaces defined by the constraints

Set of Feasible Solutions

$A = (0,0,0)^T$

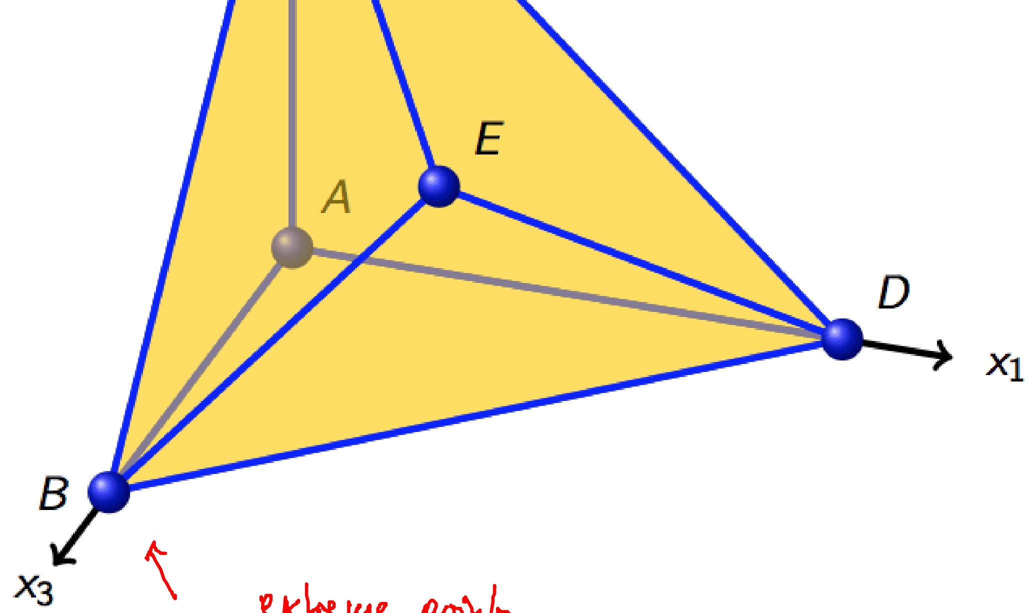


$B = (0,0,10)^T$

$C = (0,10,0)^T$

$D = (10,0,0)^T$

$E = (4,4,4)^T$



extreme points

$\hat{=}$ vertices

$\hat{=}$ basic (feasible) solutions

If objective is bounded from below
$\Rightarrow$ $\exists$ extreme point that is optimal $\Bigg\}$ $\Rightarrow$ we can try to construct an algo that investigates only extreme points

Simplex algo
works algebraically
uses <u>standard form</u>
$Ax = b, \quad x \geq 0$

needs the interpretation of extreme points as basic (feasible) solution

in example: which n lin. ind. constraints are active at a basic feas. solution?

$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ active lin. ind. constraints are $x_1 = 0$, $x_2 = 0$ $x_3 = 0$

$B = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}$ active: $x_1 = 0$ $x_2 = 0$ + first 2 inequalities

4 active constraints $\Rightarrow$ have degenerate solution

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & & \\ & 1 & \end{bmatrix}$$

matrix of active constraints contains 3 lin indep rows

go to standard form of our problem
introduce slack variables

Introducing Slack Variables

$$\begin{array}{lrcrcrcr} \min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & \\ \text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & \leq & 20 \\ & 2\,x_1 & + & x_2 & + & 2\,x_3 & \leq & 20 \\ & 2\,x_1 & + & 2\,x_2 & + & x_3 & \leq & 20 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

LP in standard form

$$\begin{array}{lrcrcrcrcrcrcr} \min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & & & & & & & \\ \text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & + & x_4 & & & & & = & 20 \\ & 2\,x_1 & + & x_2 & + & 2\,x_3 & & & + & x_5 & & & = & 20 \\ & 2\,x_1 & + & 2\,x_2 & + & x_3 & & & & & + & x_6 & = & 20 \\ & & & & & & & & & & & x_1, \ldots, x_6 & \geq & 0 \end{array}$$

polyhedron lives in $\mathbb{R}^6$
but is 3-dimensional

matrix of coefficients is

$$A \quad \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 2 & 1 & & \\ 2 & 1 & 2 & & 1 & \\ 2 & 2 & 1 & & & 1 \end{bmatrix}$$

$b = Ax = A_1 x_1 + A_2 x_2 \; -- \; A_n \cdot x_n$

want to linearly combine columns of A to get rhs b

Consider bfs of the original polyhedron

$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ active: $x_1 = 0 \quad x_2 = 0 \quad x_3 = 0 \;\rightarrow$ non basic variables

others are the basis $B = \{4, 5, 6\}$ set of basic variables

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \begin{matrix} \shortparallel & & \shortparallel \\ B(1) & & B(3) \end{matrix}$

consists of m lin. ind. columns

basic matrix $B = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$B = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}$ active $x_1 = 0 \quad x_2 = 0$, first 2 ineq in original formulation

$\downarrow$

correspond to

$x_4 = 0 \quad x_5 = 0$

in standard form

need only 3 lin. ind. ones,

several choices for a basis

Basis $\{3, 2, 6\}$, alternatively $\{3, 1, 6\}$

$\downarrow$

$B \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$E = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$ first 3 ineq. are active

$\Rightarrow$ slack is 0 $\Rightarrow$ 4, 5, 6 are non basic var.

$\Rightarrow$ basis $\{1, 2, 3\}$

$B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

now consider details of simplex algo

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & & \\ 2 & 1 & 2 & & 1 & \\ 2 & 2 & 1 & & & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

$\mathcal{B} = \{3, 2, 6\}$ defines basis

express A (by column permutation) as $\left(\overbrace{A_B}^{= B} \,\vdots\, A_N \right)$

$$Ax = b \longrightarrow (A_B \mid A_N) \begin{pmatrix} x_B \\ \hline x_N \end{pmatrix} = A_B \cdot x_B + A_N \cdot x_N$$

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 2 & 0 & 0 \end{array}\right] \xrightarrow[B^{-1}]{\text{mult with}} \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & B^{-1}A_N \\ & & 1 & \end{array}\right] x = B^{-1}b$$

A_B $\quad$ A_N $\quad$ from left

$A_B = B$

$$B^{-1} = \begin{bmatrix} -1/2 & 1 & 0 \\ 1 & -1 & 0 \\ -3/2 & 1 & 1 \end{bmatrix}$$

$$x_B = B^{-1}b = \begin{pmatrix} 10 \\ 0 \\ 10 \end{pmatrix} \begin{matrix} x_3 \\ x_2 \\ x_6 \end{matrix} \qquad x_N = 0$$

$\Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 0 \\ 10 \end{pmatrix}$ feas sol. of standard form

$\Rightarrow$ (projection on original form) $\begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} = C$

Simplex algo starts in a bfs and looks at

<u>adjacent</u> bfs

(only one column different in their bases)

→ bring a non basic column into the basis

remove a basic column

for that purpose, we defined the <u>jth basic direction</u> d

$j = 1$ basic direction for $j = 1$ is

$$d = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{bmatrix} 1 \\ 1 \\ -3/2 \\ 0 \\ 0 \\ -5/2 \end{bmatrix} \begin{array}{l} \\ \leftarrow \\ \leftarrow \text{ change} \\ \text{in basic} \\ \leftarrow \text{ variables} \\ \\ \end{array} \qquad d_B = -B^{-1}A_j \underset{\text{Expl.}}{\overset{\uparrow}{=}} \begin{pmatrix} -3/2 \\ 1 \\ -5/2 \end{pmatrix} \begin{array}{c} x_3 \\ x_2 \\ x_6 \end{array}$$

have shown $A(x + \Theta d) = b$ (i.e. $Ad = 0$)

<u>reduced cost</u> (always 0 for basic variables)

non basic variable : $\bar{c}_j \overset{\text{Def}}{=} c_j - c_B^T B^{-1} A_j$

$= c_j - c_B^T d$

can improve the objective value by taking A_j into basis

if $\bar{c}_j < 0$

$j = 1 \qquad \bar{c}_1 = -10 - (-12, -12, 0)\, B^{-1} A_1 \quad = -4 < 0$

So we want to go from x to $x + \Theta d$ for some $\Theta \geq 0$

we know that equations are still fulfilled $A(x+\Theta d) = b$

so only need to check $x + \Theta d \geq 0$

$$\Rightarrow \quad \Theta^* = \min_{\substack{l=1,\dots,m \\ d_{B(l)} < 0}} \frac{-x_{B(l)}}{d_{B(l)}}$$

for $j=1$: $\Theta^* = \min \left\{ \underset{x_3}{\frac{10}{3/2}}, \underset{x_6}{\frac{10}{5/2}} \right\} = 4$ for $B(l) = 6$

$\Rightarrow$ variable 6 leaves the basis when variable 1 enters
column column

new basis $\bar{B} = \{3, 2, 1\}$ $\quad \bar{B} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

$$\bar{B}^{-1} = \begin{bmatrix} 2 & 2 & -3 \\ 2 & -3 & 2 \\ -3 & 2 & 2 \end{bmatrix} \cdot 1/5$$

$x_{\bar{B}} = \bar{B}^{-1} b = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \begin{matrix} x_3 \\ x_2 \\ x_1 \end{matrix}$ $\hat{=}$ vertex E in original polytope

$\bar{c}_j = c_j - c_{\bar{B}}^T \bar{B}^{-1} A_j \geq 0 \quad j \in \bar{N}$ $\Rightarrow$ $\boxed{\text{we are optimal}}$
Thm 3.6

Expl $\bar{c}_4 = 0 - (-12, -12, -10) \, \bar{B}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$= (12, 12, 10) \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} > 0$