

Chapter 4: Duality Theory

(cp. Bertsimas & Tsitsiklis, Chapter 4)

Motivation

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, consider the linear program

$$\min c^T \cdot x \quad \text{s.t.} \quad A \cdot x \geq b, x \geq 0$$

Question: How to derive lower bounds on the optimal solution value?

Idea: For $p \in \mathbb{R}^m$ with $p \geq 0$: $A \cdot x \geq b \implies (p^T \cdot A) \cdot x \geq p^T \cdot b$

Thus, if $c^T \geq p^T \cdot A$, then

$$c^T \cdot x \geq (p^T \cdot A) \cdot x \geq p^T \cdot b \quad \text{for all feasible solutions } x.$$

Find the best (largest) lower bound in this way:

$$\begin{array}{ll} \max & p^T \cdot b \\ \text{s.t.} & p^T \cdot A \leq c^T \\ & p \geq 0 \end{array} \quad \longleftrightarrow \quad \begin{array}{ll} \max & b^T \cdot p \\ \text{s.t.} & A^T \cdot p \leq c \\ & p \geq 0 \end{array}$$

This LP is the dual linear program of our initial LP.

Primal and Dual Variables and Constraints

primal LP (minimize)		dual LP (maximize)	
	$\geq b_i$	≥ 0	
constraints	$\leq b_i$	≤ 0	variables
	$= b_i$	free	
	≥ 0	$\leq c_j$	
variables	≤ 0	$\geq c_j$	constraints
	free	$= c_j$	

is used in the motivation

Basic Properties of the Dual Linear Program

Theorem 4.1.

The dual of the dual LP is the primal LP.

Proof:...

Theorem 4.2.

Let Π_1 and Π_2 be two LPs where Π_2 has been obtained from Π_1 by (several) transformations of the following type:

- i** replace a free variable by the difference of two non-negative variables;
- ii** introduce a slack variable in order to replace an inequality constraint by an equation;
- iii** if some row of a feasible equality system is a linear combination of the other rows, eliminate this row.

Then the dual of Π_1 is equivalent to the dual of Π_2 .

Proof:...

Proof of Thm 4.1: Assume primal is of general form with $=$ and \geq constraints

Write the dual in primal form:

$$\begin{aligned} \min \pi^T(-b) \quad & \text{such that} \\ (-A_j^T)\pi & \geq -c_j \quad j \in N \\ (-A_j^T)\pi & = -c_j \quad j \in \bar{N} \\ \pi_i & \geq 0 \quad j \in \bar{M} \\ \pi_i & \text{ unconstrained} \quad j \in M \end{aligned}$$

$$\pi \stackrel{!}{=} p$$

primal

$$\text{with } c^T x$$

$$a_i^T x = b_i \quad i \in \bar{K}$$

$$a_i^T x \geq b_i \quad i \in K$$

$$x_j \geq 0 \quad j \in N$$

$$x_j \text{ free} \quad j \in \bar{N}$$

The transformation rules yield the following dual LP

$$\begin{aligned} \max x^T(-c) \quad & \text{such that} \\ x_j & \geq 0 \quad j \in N \\ x_j & \text{ unconstrained} \quad j \in \bar{N} \\ -a_i^T x & \leq -b_i \quad i \in \bar{M} \\ -a_i^T x & = -b_i \quad i \in M \end{aligned}$$

which is the primal LP \square

Proof of Thm 4.2

(i) splitting variables

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax \geq b \\ x \text{ free} \end{aligned}$$

$$\begin{aligned} \xrightarrow{\text{transform}} \\ x = x^+ - x^- \\ x^+, x^- \geq 0 \end{aligned}$$

$$\begin{aligned} \text{with } c^T x^+ - c^T x^- \\ \text{s.t. } Ax^+ - Ax^- \geq b \\ x^+, x^- \geq 0 \end{aligned}$$

\Downarrow

$$\text{with } \begin{pmatrix} c \\ -c \end{pmatrix}^T \begin{pmatrix} x^+ \\ x^- \end{pmatrix}$$

$$\text{s.t. } (A \mid -A) \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \geq b$$

$$\begin{aligned} \text{dualize } \downarrow \\ \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \geq 0 \end{aligned}$$

\downarrow dualize

$$\begin{array}{l}
 \max p^T b \\
 \text{s.t. } \boxed{p^T A = c^T} \\
 p \geq 0
 \end{array}
 \quad \longleftrightarrow \quad \text{equivalent} \quad
 \begin{array}{l}
 \max p^T b \\
 \text{s.t. } \boxed{p^T (A \mid -A) \geq \begin{pmatrix} c \\ -c \end{pmatrix}} \\
 p \geq 0
 \end{array}$$

(ii) adding slack variables

$$\begin{array}{l}
 \text{min } c^T x \\
 \text{s.t. } Ax \geq b \\
 x \text{ free}
 \end{array}
 \quad \xrightarrow[\text{add slack variables}]{\text{transform}} \quad
 \begin{array}{l}
 \text{min } c^T x + 0^T y \\
 \text{s.t. } Ax - Iy = b \\
 y \geq 0
 \end{array}$$

$$\begin{array}{l}
 \max p^T b \\
 \text{s.t. } p^T A = c^T \\
 \boxed{p \geq 0}
 \end{array}
 \quad \longleftrightarrow \quad \text{equiv.} \quad
 \begin{array}{l}
 \max p^T b \\
 p^T A = c^T \\
 \boxed{p^T (-I) \leq 0^T} \\
 p \text{ free}
 \end{array}$$

(iii) eliminating redundant rows

$$\begin{array}{l}
 \text{min } c^T x \\
 \text{s.t. } Ax = b \\
 x \geq 0
 \end{array}$$

assume row m is a linear combination of rows $1, \dots, m-1$

$$\text{i.e. } \boxed{
 \begin{aligned}
 a_m^T &= \sum_{i=1}^{m-1} \gamma_i a_i^T \\
 b_m &= \sum_{i=1}^{m-1} \gamma_i b_i
 \end{aligned}
 } \quad (*)$$

↓ dualize

$$\max \sum_{i=1}^k p_i \cdot b_i \stackrel{(*)}{=} \sum_{i=1}^{k-1} (p_i + \gamma_i p_m) b_i$$

$$\text{s.t. } \sum_{i=1}^k p_i a_i^T \leq c^T \xrightarrow{(*)} \sum_{i=1}^{k-1} \underbrace{(p_i + \gamma_i p_m)}_{=: q_i} a_i^T \leq c^T$$

$$\Rightarrow \max \sum_{i=1}^{k-1} q_i b_i$$

$$\text{s.t. } \sum_{i=1}^{k-1} q_i a_i^T \leq c^T$$

} dual of original problem
with last row deleted

Weak Duality Theorem

Theorem 4.3.

If x is a feasible solution to the primal LP (minimization problem) and p a feasible solution to the dual LP (maximization problem), then

$$c^T \cdot x \geq p^T \cdot b .$$

Proof:...

Corollary 4.4.

Consider a primal-dual pair of linear programs as above.

- a** If the primal LP is unbounded (i. e., optimal cost = $-\infty$), then the dual LP is infeasible.
- b** If the dual LP is unbounded (i. e., optimal cost = ∞), then the primal LP is infeasible.
- c** If x and p are feasible solutions to the primal and dual LP, resp., and if $c^T \cdot x = p^T \cdot b$, then x and p are optimal solutions.

clear

→ Proof

Theorem 4.2 \Rightarrow may assume that primal (P) and dual (D) are of the form

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$(D) \quad \begin{array}{ll} \max & \pi^T b \\ \text{s.t.} & p^T A \leq c^T \\ & p \geq 0 \end{array}$$

Let x, p be feasible for (P), (D) Then

$$c^T x \geq (p^T A) x = p^T (Ax) \underset{Ax=b}{=} p^T b \quad \square$$

Strong Duality Theorem

Theorem 4.5.

If an LP has an optimal solution, so does its dual and the optimal costs are equal.

Proof:...

dual (dual) = primal \Rightarrow may assume that the primal (P) has an optimal solution

Thm. 4.2 \Rightarrow may assume that (P) is in standard form

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

optimum attained at some bfs

\Rightarrow may assume that optimal solution y is a bfs with basis B

y optimal \Rightarrow reduced cost $\boxed{\bar{c}^T = c^T - \underbrace{c_B^T B^{-1} A}_{=: q^T} \geq 0} \quad (*)$

(D) $\max p^T b$
 s.t. $p^T A \leq c^T$
 p free

\Downarrow
 $q^T A \leq c^T$
 $\Rightarrow q$ is feasible in (D)

Then $q^T b = (c_B^T B^{-1}) b = c_B^T (B^{-1} b) = c_B^T y_B = c^T y$

values of basic variables in y
 non basic $y_j = 0$

$\Rightarrow q^T b = c^T y$

$\Rightarrow q$ is optimal in (D) \square

Different Possibilities for Primal and Dual LP

primal \ dual	finite optimum	unbounded	infeasible
finite optimum	strong duality possible	impossible	impossible
unbounded	impossible	impossible	possible ?
infeasible	impossible	possible ?	possible ?

partly exercises

Example of infeasible primal and dual LP:

$\min x_1 + 2x_2$

$\max p_1 + 3p_2$

$$\text{s.t. } x_1 + x_2 = 1$$

$$2x_1 + 2x_2 = 3$$

$$\text{s.t. } p_1 + 2p_2 = 1$$

$$p_1 + 2p_2 = 2$$

Complementary Slackness

Consider the following pair of primal and dual LPs:

$$\min c^T \cdot x$$

$$\text{s.t. } A \cdot x \geq b$$

$$\max p^T \cdot b$$

$$\text{s.t. } p^T \cdot A = c^T$$

$$p \geq 0$$

If x and p are feasible solutions, then $c^T \cdot x = p^T \cdot A \cdot x \geq p^T \cdot b$. Thus,

$$c^T \cdot x = p^T \cdot b \iff \text{for all } i: p_i = 0 \text{ if } a_i^T \cdot x > b_i.$$

Theorem 4.6.

Consider an arbitrary pair of primal and dual LPs. Let x and p be feasible solutions to the primal and dual LP, respectively. Then x and p are both optimal if and only if

$$(1) \quad u_i := p_i \overbrace{(a_i^T \cdot x - b_i)}^{\text{primal slack}} = 0 \quad \text{for all } i, \quad (\text{dual variable}) \cdot (\text{primal slack})$$

$$(2) \quad v_j := \underbrace{(c_j - p^T \cdot A_j)}_{\text{dual slack}} x_j = 0 \quad \text{for all } j. \quad (\text{dual slack}) \cdot (\text{primal var.})$$

Proof:...

$$u_i \geq 0$$

since

$$a_i^T x - b_i = 0 \Rightarrow u_i = 0$$

$$a_i^T x - b_i \geq 0 \Rightarrow \pi_i \geq 0 \Rightarrow u_i \geq 0$$

$$v_j \geq 0$$

since

$$x_j \text{ unconstrained} \Rightarrow \pi^T A_j = c_j \Rightarrow v_j = 0$$

$$x_j \geq 0 \Rightarrow \pi^T A_j \leq c_j \Rightarrow v_j \geq 0$$

with dual variables π_i instead of p_i

● Set $u := \sum_i u_i$, $v := \sum_j v_j \Rightarrow u, v \geq 0$. Then

$u = 0 \Leftrightarrow (1)$ holds

$v = 0 \Leftrightarrow (2)$ holds

● Then

$$\begin{aligned} u + v &= \sum_i \pi_i (a_i^T x - b_i) + \sum_j (c_j - \pi^T A_j) x_j \\ &= -\sum_i \pi_i b_i + \sum_j c_j x_j + \sum_i \pi_i a_i^T x - \sum_j \pi^T A_j x_j \\ &= -\pi^T b + c^T x + (\pi^T A)x - \pi^T (Ax) \\ &= -\pi^T b + c^T x \end{aligned}$$

Hence: $u + v = -\pi^T b + c^T x$

● Suppose (1) and (2) hold $\Rightarrow u + v = 0 \Rightarrow c^T x = \pi^T b$

Weak Duality Theorem $\Rightarrow x, \pi$ are optimal

● Suppose that x and π are optimal

Strong Duality Theorem $\Rightarrow c^T x = \pi^T b \Rightarrow u + v = 0 \Rightarrow (4.8)$ and $(4.9) \quad \square$