Modul Prüfung

jetzt 90 Teilnehmer $\Rightarrow$ Klausur

Terminvorschläge: 1. Klausur 6./7. März

2. Klausur 7./8. April

Achtung: • wer eine Klausur ohne wichtigen Grund versäumt und bei der 2. durchfällt muss auf die Prüfungen des nächsten Zyklus warten (dann gilt auch der Stoff des " ")

• mündliche Prüfungen nur bei 2. Wh. oder aus wichtigen vom Prüfungsausschuss zu genehmigten Gründen

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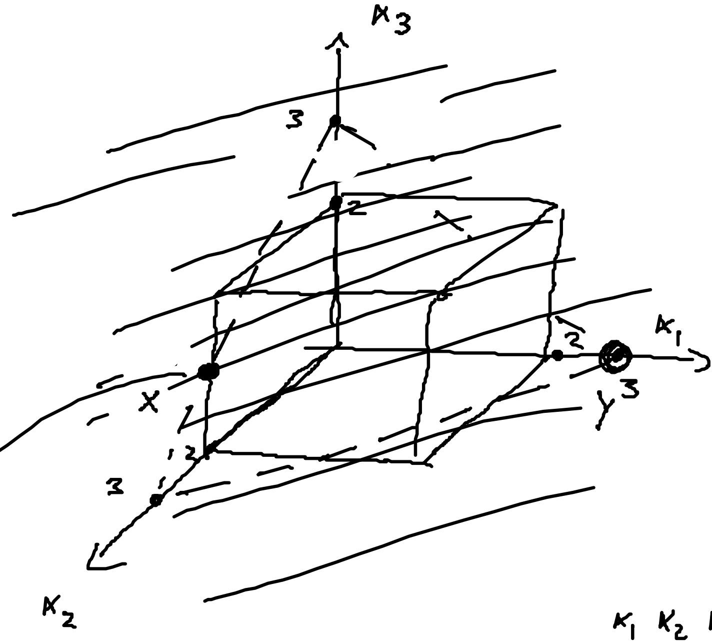
Def 2.10: Equations, Inequalities $\Big\}$ are given, define the polyhedron P

basic solution $\overset{\text{Def}}{\Leftarrow}$ all equations satisfied (i.e. active)

n lin. ind. constraints are active

x basic feasible solution, if, in addition, $x \in P$

Expl.



(1) $x_1 + x_2 + x_3 = 3$

(2) $x_1 \geq 0$

(3) $x_2 \geq 0$

(4) $x_3 \geq 0$

(5) $-x_1 \geq -2$

(6) $-x_2 \geq -2$

(7) $-x_3 \geq -2$

coefficient

$x \in P$ active constraints:

$$\begin{array}{c} \\ (1) \\ (6) \\ (2) \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \end{array}$$

linearly indep rows

$\Rightarrow$ basic feasible solution (bfs)

$y \notin P$ active constraints

$$\begin{array}{c} (1) \\ (3) \\ (4) \end{array} \begin{pmatrix} 1 & 1 & 1 \\ & 1 & \\ & & 1 \end{pmatrix}$$

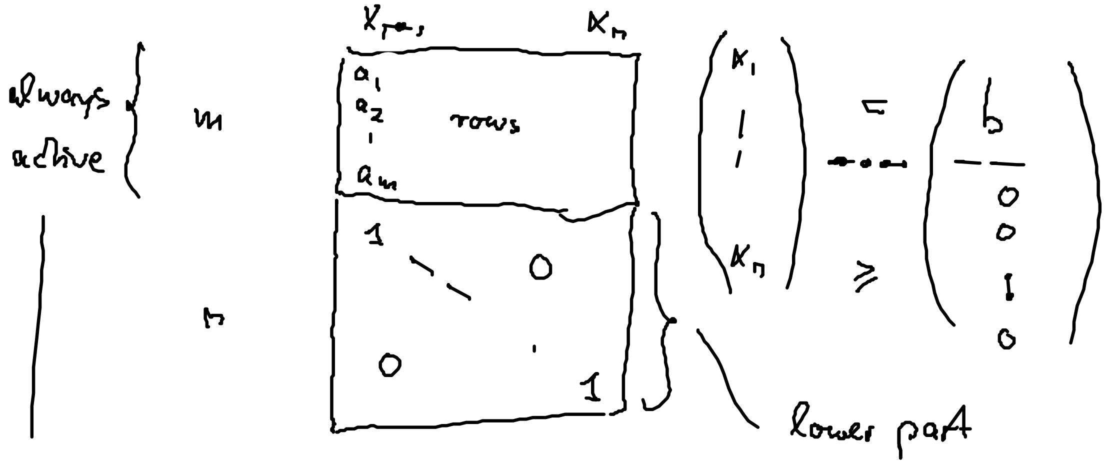
lin indep rows

basic solution
not feasible

Simplex also works on bfs and uses polyhedra in standard form

$Ax = b$ $\quad$ rank$(A) = m$

$x \geq 0$



and lin. indep., can always choose the remaining $n-m$ active constraints (in a bfs) from the lower part

because: upper part (rows of A) span a m-dimensional subspace of $\mathbb{R}^n$

$\Rightarrow$ lower part contains a unique selection of rows that span the orthogonal complement to $\mathrm{span}\{a_1, \dots, a_m\}$

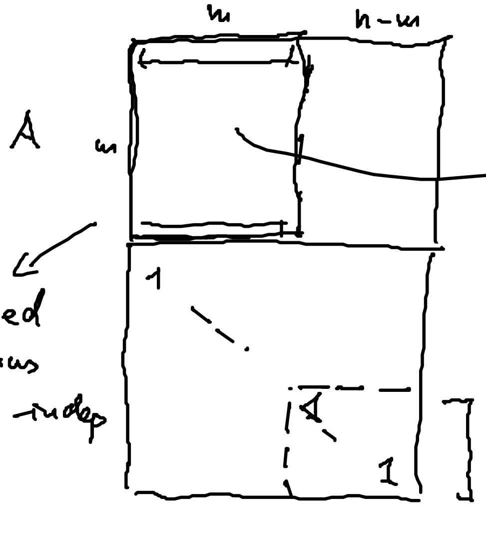
$\Rightarrow$ these must <u>be active</u>, because we started with n lin. ind. active constraints

$\Rightarrow$ these variables are 0

they are calle the <u>non basic variables</u>

the others are called <u>basic variables</u>

and denoted by $B(1), B(2) \dots B(m)$)



basis matrix

shortened rows must be lin-indep

$\Downarrow$

assume these are active

$\Rightarrow$ these are the <u>non-basic variables</u>

columns $B(1), \ldots, B(m)$ are lin. indep.

Basis 1:

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

B(1) = 4, B(2) = 5, B(3) = 6, B(4) = 7

Basis 2

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

B(1) = 2, B(2) = 5, B(3) = 6, B(4) = 7

Basis 3

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

B(1) = 2, B(2) = 1, B(3) = 3, B(4) = 7

$$B = \begin{pmatrix} 1 & 1 & 1 & \\ & 1 & & \\ & & 1 & \\ 3 & & 1 & 1 \end{pmatrix}$$

here we go the other way: choose m lin. ind. columns from A that define the basis variables, the others are the non-basic variables

rows of A + ~~rows of~~ these non-basic variable are n lin ind. active constraints

$\Rightarrow$ this defines a basic solution

$\Rightarrow \exists\, x_1, \ldots, x_4$ s.t.

$$A_{B(1)} \cdot x_1 + \ldots + A_{B(4)} x_4 = b$$

or $B \; x_B = b \Rightarrow x_B = B^{-1} b$

$$B = \left(A_{B(1)} \; \cdots \; A_{B(m)}\right)$$

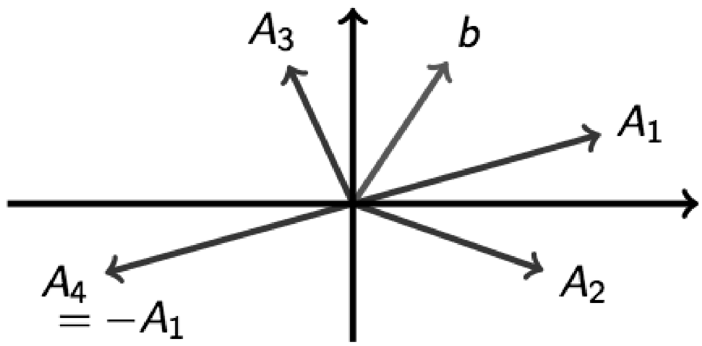
Basic Columns and Basic Solutions

Observation 2.15.

Let $x \in \mathbb{R}^n$ be a basic solution, then:

- $B \cdot x_B = b$ and thus $x_B = B^{-1} \cdot b$;
- x is a basic feasible solution if and only if $x_B = B^{-1} \cdot b \geq 0$.

Example: $m = 2$



- A_1, A_3 or A_2, A_3 form bases with corresp. basic feasible solutions.
- A_1, A_4 do not form a basis.
- A_1, A_2 and A_2, A_4 and A_3, A_4 form bases with infeasible basic solution.

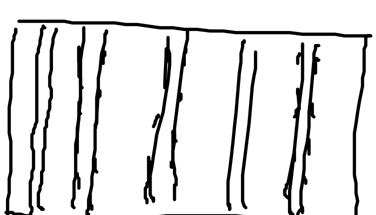
Bases and Basic Solutions

Corollary 2.16.

- Every basis $A_{B(1)}, \ldots, A_{B(m)}$ determines a unique basic solution.
- Thus, different basic solutions correspond to different bases.
- But: two different bases might yield the same basic solution.

Example: If $b = 0$, then $x = 0$ is the only basic solution.

A



blue $\hat{=}$ B

red $\hat{=}$ B'

Adjacent Bases

Definition 2.17.

Two bases $A_{B(1)}, \ldots, A_{B(m)}$ and $A_{B'(1)}, \ldots, A_{B'(m)}$ are adjacent if they share all but one column.

Observation 2.18.

a. Two adjacent basic solutions can always be obtained from two adjacent bases.
b. If two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

Degeneracy

Definition 2.19.

A basic solution x of a polyhedron P is degenerate if more than n constraints are active at x.

Observation 2.20.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ be a polyhedron in standard form with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

a. A basic solution $x \in P$ is degenerate if and only if more than $n - m$ components of x are zero.
b. For a non-degenerate basic solution $x \in P$, there is a unique basis.

Three Different Reasons for Degeneracy

i redundant variables

Example:

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_3 &= 0 \\ x_1,\, x_2,\, x_3 &\geq 0 \end{aligned} \quad \longleftrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

only basis

$x_2 = 1$

$x_3 = 0$

non basic var. $x_1 = 0$

more than $n-m$ variables are 0

ii redundant constraints

Example:

$$\begin{aligned} x_1 + 2x_2 &\leq 3 \\ 2x_1 + x_2 &\leq 3 \\ x_1 + x_2 &\leq 2 \\ x_1,\, x_2 &\geq 0 \end{aligned}$$

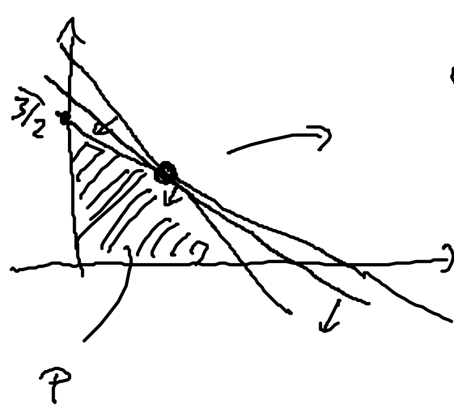
iii geometric reasons

Example: Octahedron

Observation 2.21.

Perturbing the right hand side vector b may remove degeneracy.

(ii)



active constraints

1	2
2	1
1	1

degenerate

Existence of Extreme Points

Definition 2.22.

A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

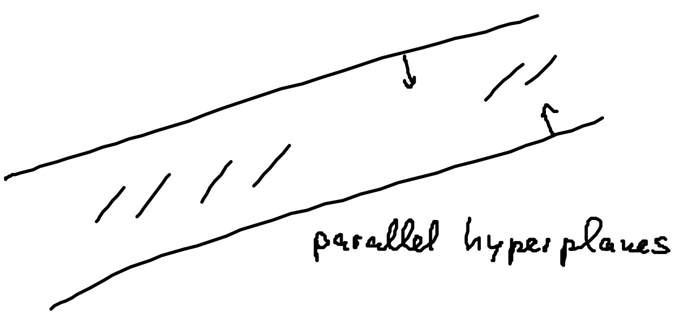
$$x + \lambda \cdot d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

Theorem 2.23.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\} \neq \emptyset$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

1. There exists an extreme point $x \in P$.
2. P does not contain a line.
3. A contains n linearly independent rows.

Proof: ...



P contains a line but no extreme point

Proof of Thm 2.23

$\underline{(i) \Rightarrow (iii)}$ $\quad \underline{\exists \text{ extreme point } x \Rightarrow n \text{ lin. ind. rows}}$

$\Downarrow$ 2.11

x bfs

$\underline{(iii) \Rightarrow (ii)}$ $\quad \underline{n \text{ lin indep rows} \Rightarrow \text{no line}}$

assume that P contains a line, P given as $Ax \geq b$

$\Rightarrow \exists\, x \in P,\ d \neq 0$ s.t. $x + \lambda \cdot d \in P \quad \forall \lambda \in \mathbb{R}$

$\Rightarrow A(x+\lambda d) = Ax + \lambda A d \geq b \quad \forall \lambda$

$\Rightarrow Ad = 0 \Rightarrow \underbrace{d_1 \cdot A_1 + d_2 A_2 + \dots + d_n A_n}_{\text{columns of } A} = 0$

$\Rightarrow (\text{rank } A = n) \Rightarrow d = 0$, contradiction

<u>(ii) $\Rightarrow$ (i)</u> <u>no line $\Rightarrow$ extreme point</u>

$\uparrow$
show this by finding n lin. ind. active constraints

Let $x \in P$ s.t. $|\{i \mid a_i^T x = b_i\}|$ is maximum

show that I contains n lin. ind. constraints

assume not

$\Rightarrow \exists d \in \mathbb{R}^n \setminus \{0\}$ s.t. $a_i^T d = 0 \quad \forall i \in I$

$\Rightarrow a_i^T (x + \lambda d) = b_i \quad \forall \lambda \quad \forall i \in I$

no line in $P \Rightarrow x + \lambda d$ violates some constraint for $|\lambda|$ large
say i_0

$\Rightarrow$ for some λ_0 $\quad a_{i_0}^T (x + \lambda d) \begin{cases} = b_{i_0} \\ < & \text{for } \lambda_0 - \varepsilon \\ > & \text{for } \lambda_0 + \varepsilon \end{cases}$

$\Rightarrow i_0 \notin I$

$\Rightarrow$ have found a new active constraint, contradiction to choice of I $\square$

Existence of Extreme Points (cont.)

Corollary 2.24.

- **a** A non-empty polytope contains an extreme point.
- **b** A non-empty polyhedron in standard form contains an extreme point.

Proof of b:

$$\begin{aligned} A \cdot x &= b \\ x &\geq 0 \end{aligned} \qquad \longleftrightarrow \qquad \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \cdot x \geq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$

□

Example:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrl} x_1 & + & x_2 & \geq 1 \\ x_1 & + & 2\,x_2 & \geq 0 \end{array} \right\}$$

defines a set in $x_1 - x_2$ plane

contains a line since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in P$ for all $\lambda \in \mathbb{R}$.