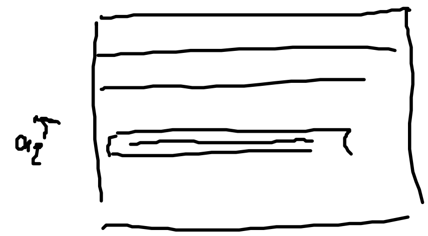
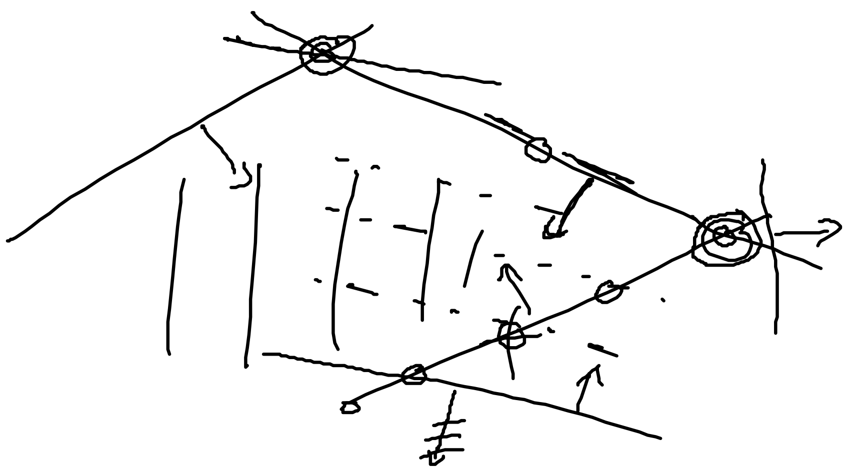
$\{ x \in \mathbb{R}^n \mid Ax \geq b \}$ Polyhedron P

A  $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

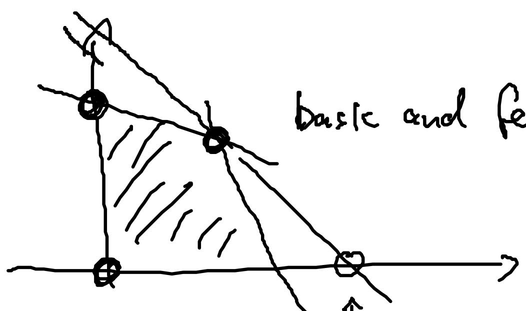
rows of A define halfspaces



- extremal point $\in P$ $:=$ not a convex comb. of 2 other points
- vertex $x \in P$ $:= \exists\, c \in \mathbb{R}^n$ s.t. $c^T x < c^T y \;\; \forall y \in P$
  $\iff$ x is unique solution of $\min \{ c^T y \mid y \in P \}$
- basic (feasible) solution

  equality constraints

  m inequality —"—



basic and feasible (bfs)

basic, but not feasible

Proof of Theorem 2.11

(i) $\Rightarrow$ (ii) Excercise

(ii) $\Rightarrow$ (iii)

Let x^* be an extreme point (not a convex comb of 2 other points)

Show that x^* is a basic feasible solution

i.e. $x^* \in P$, all equality constraints are active
(a) (b)

$\exists$ n linearly indep. active constraints at x^*
(c)

Assume not $\Rightarrow$ one of (a) (b) (c) must be wrong

(a) fulfilled, because $x^* \in P$ as extrem point

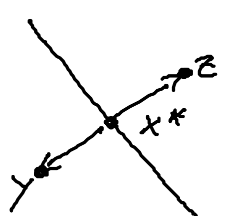
(b) -"- because $x^* \in P \Rightarrow$ all equations hold

(c) less than n linearly ind. vectors in $\{ a_i \mid i \in I \}$

$\uparrow$ set of constraints active at x^*

$\Rightarrow \exists\, d \in \mathbb{R}^n$ and $a_i^T d = 0 \quad \forall i \in I$

(a_i do not span $\mathbb{R}^n$)



span of $\{ a_i \mid i \in I \}$ in $\mathbb{R}^2$

For $\varepsilon > 0$ consider $y := x^* + \varepsilon d$

$z := x^* - \varepsilon d$

$\Rightarrow x^* = \frac{1}{2} y + \frac{1}{2} z \Rightarrow$ contradiction if $y, z \in P$

(to extreme point)

show this

for $i \in I$: $a_i^T y = \underbrace{a_i^T x^*}_{= b_i} + \varepsilon \cdot \underbrace{a_i^T d}_{0} = b_i$

$a_i^T z = b_i$

$i \notin I$: $a_i^T x^* > b_i$

↑

Polyhedron defined by $Ax \geq b$

$i \notin I$ not active at x^*

$a_i^T y = \underbrace{a_i^T x^*}_{> b_i} + \varepsilon \underbrace{a_i^T d}_{\text{maybe} \neq 0} \geq b_i$ for ε small

$a_i^T z \geq b_i$ – " –

$\Rightarrow z, y \in P \Rightarrow$ contradiction

(iii) $\Rightarrow$ (i) Let $I = \{ i \mid a_i^T x^* = b_i \}$

bfs vertex

need to define an objective c, s.t. x^* is unique min. of $\min \{ c^T x \mid x \in P \}$

choose $c := \sum_{i \in I} a_i$

$\Rightarrow c^T x^* = \sum_{i \in I} \underbrace{a_i^T x^*}_{= b_i} = \sum_{i \in I} b_i$

arbitrary $x \in P$: $c^T x = \sum_{i \in I} \underbrace{a_i^T x}_{\geq b_i} \geq \sum_{i \in I} b_i$

$\Rightarrow$ x^* is an optimal solution for the chosen c

Let x be another optimal solution

$\Rightarrow$ $c^T x = \sum_{i \in I} b_i$ $\Rightarrow$ $a_i^T x = b_i$ $\forall i \in I$

$a_i^T x \geq b_i$

x^* is a b.f.s. $\Rightarrow$ I contains n linearly indep. rows

$\Rightarrow$ $x = x^*$ (system $a_i^T x = b_i$ has a unique solution) $\square$

Number of Vertices

Corollary 2.12.

a) A polyhedron has a finite number of vertices and basic solutions.

b) For a polyhedron in $\mathbb{R}^n$ given by linear equations and m linear inequalities, this number is at most $\binom{m}{n}$.

Example:

$P := \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \ i = 1, \dots, n\}$ (n-dimensional unit cube)

- number of constraints: $m = 2n$
- number of vertices: 2^n

k | equality constraints } may assume that these are lin. indep. (others are redundant)

m | Inequ. constraints } can choose only $n-k$ here so at most $\binom{m}{n}$

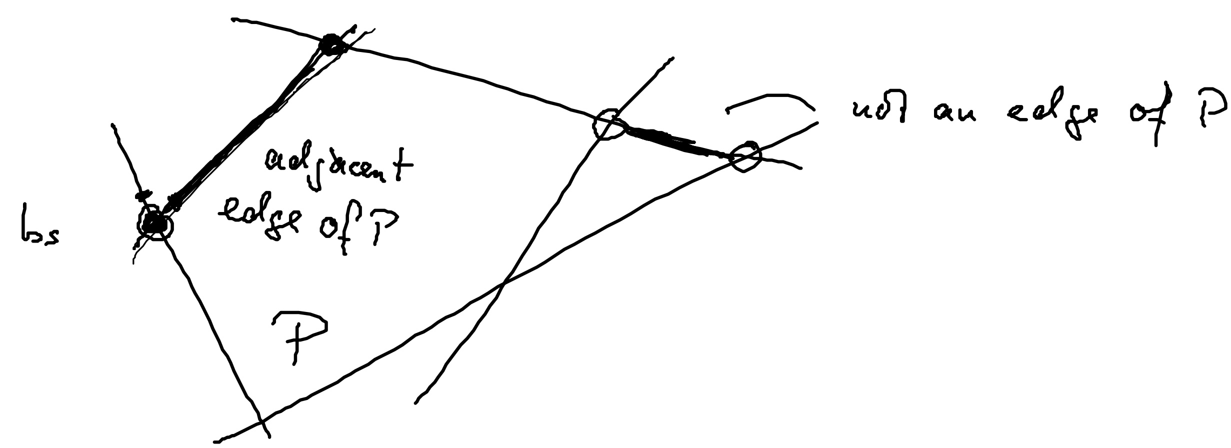
$$\binom{m}{n} = \binom{2n}{n} = \frac{2n(2n-1)\dots}{n\ (n-1)\ \dots} \geq 2^n$$

Adjacent Basic Solutions and Edges

Definition 2.13.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a. Two distinct basic solutions are adjacent if there are $n-1$ linearly independent constraints that are active at both of them.
b. If both solutions are feasible, the line segment that joins them is an edge of P.



Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$.

Observation

One can assume without loss of generality that $\text{rank}(A) = m$.

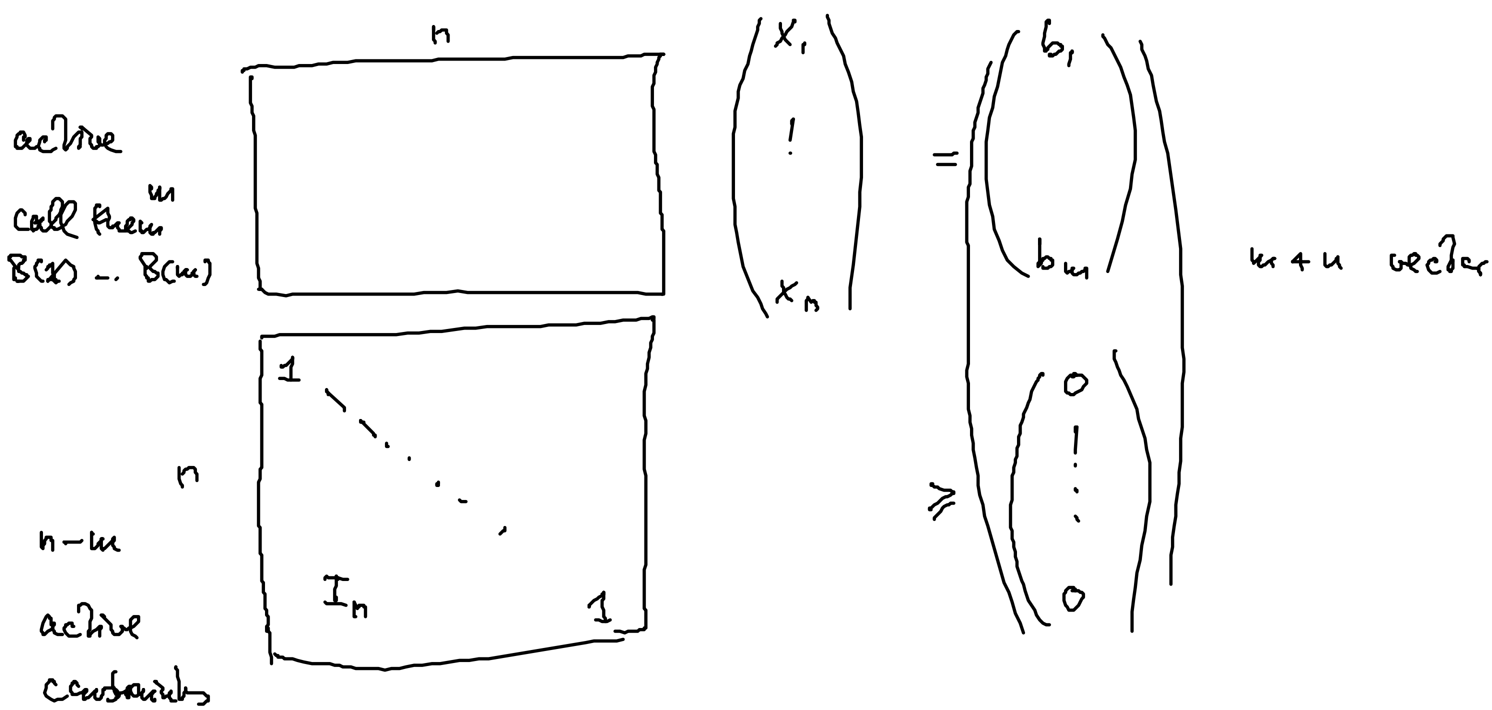
Theorem 2.14.

$x \in \mathbb{R}^n$ is a basic solution of P if and only if $A \cdot x = b$ and there are indices $B(1), \dots, B(m) \in \{1, \dots, n\}$ such that

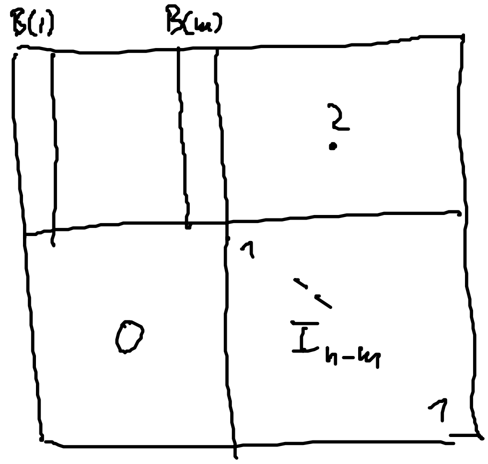
- columns $A_{B(1)}, \dots, A_{B(m)}$ of matrix A are linearly independent and
- $x_i = 0$ for all $i \notin \{B(1), \dots, B(m)\}$.

Proof: ...

- $x_{B(1)}, \dots, x_{B(m)}$ are basic variables, the remaining variables non-basic.
- The vector of basic variables is denoted by $x_B := (x_{B(1)}, \dots, x_{B(m)})^T$.
- $A_{B(1)}, \dots, A_{B(m)}$ are basic columns of A and form a basis of $\mathbb{R}^m$.
- The matrix $B := (A_{B(1)}, \dots, A_{B(m)}) \in \mathbb{R}^{m \times m}$ is called basis matrix.



w.l.o.g. $B(i) = i$



matrix for these constraints

3.3 Example

$$
\begin{array}{lccccccccc}
\min & 2x_1 & & & +x_4 & & & +5x_7 & \\
\text{s.t.} & x_1 & +x_2 & +x_3 & +x_4 & & & & =4 \\
 & x_1 & & & & +x_5 & & & =2 \\
 & & & +x_3 & & & +x_6 & & =3 \\
 & & +3x_2 & +x_3 & & & & +x_7 & =6 \\
 & x_j & \geq 0 & \forall j & & & & &
\end{array}
$$

$$
(A|b) = \left(\begin{array}{ccccccc|c}
1 & 1 & 1 & 1 & & & & 4 \\
1 & & & & 1 & & & 2 \\
 & & 1 & & & 1 & & 3 \\
 & 3 & 1 & & & & 1 & 6
\end{array}\right)
$$

A has full row rank $m = 4$

Basis 1:

$$
\left(\begin{array}{ccccccc|c}
1 & 1 & 1 & \boxed{1} & & & & 4 \\
1 & & & & \boxed{1} & & & 2 \\
 & & 1 & & & \boxed{1} & & 3 \\
 & 3 & 1 & & & & \boxed{1} & 6
\end{array}\right)
$$

B(1) = 4, B(2) = 5, B(3) = 6, B(4) = 7

- Basis 2

$$\left(\begin{array}{ccccccc|c} 1 & \boxed{1} & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

B(1) = 2, B(2) = 5, B(3) = 6, B(4) = 7

- Basis 3

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right)$$

B(1) = 2, B(2) = 1, B(3) = 3, B(4) = 7

$$B = \begin{pmatrix} 1 & 1 & 1 & \\ & 1 & & \\ & & 1 & \\ 3 & & 1 & 1 \end{pmatrix}$$

- Every basis matrix is invertible and can be transformed into the identity matrix by elementary row operations and column permutations (Gaussian elimination).
- If we transform the whole extended matrix (A|b) with these operations, we obtain a solution of Ax = b by setting the basic variables to the (transformed) right hand side, and the non-basic variables to 0. This solution is called the basic solution for basis B.
  - The applet below can be used to carry out these operations
  - http://people.hofstra.edu/faculty/Stefan_Waner/RealWorld/tutorialsf1/scriptpivot2.html

3.3 Example (continued)

Basis 1

no transformation needed since B = identity matrix

basic solution: $x_4 = 4, x_5 = 2, x_6 = 3, x_7 = 6, x_j = 0$ otherwise

Basis 2

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right) \Rightarrow \left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ -3 & & -2 & -3 & & & 1 & -6 \end{array}\right)$$

B(1) = 2, B(2) = 5, B(3) = 6, B(4) = 7

basic solution: $x_2 = 4, x_5 = 2, x_6 = 3, x_7 = -6, x_j = 0$ otherwise

Basis 3

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & & & & 4 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & 3 & 1 & & & & 1 & 6 \end{array}\right) \Rightarrow \left(\begin{array}{ccccccc|c} & 1 & & 1 & -1 & -1 & & -1 \\ 1 & & & & 1 & & & 2 \\ & & 1 & & & 1 & & 3 \\ & & & -3 & 3 & 2 & 1 & 6 \end{array}\right)$$

B(1) = 2, B(2) = 1, B(3) = 3, B(4) = 7

basic solution: $x_2 = -1, x_1 = 2, x_3 = 3, x_7 = 6, x_j = 0$ otherwise

If we permute the columns of A and x such that $A = (A_B, A_N)$ and $x = (x_B, x_N)^T$, then the elementary transformations correspond to multiplying the linear system $(A_B, A_N)(x_B, x_N)^T = b$ from the left with the inverse B^{-1} of the basis:

$$B^{-1}(A_B, A_N)(x_B, x_N)^T = B^{-1}b$$

$$\Leftrightarrow \quad B^{-1}A_Bx_B + B^{-1}A_Nx_N = B^{-1}b$$

$$\Leftrightarrow \quad x_B + B^{-1}A_Nx_N = B^{-1}b$$

If we set $x_N = 0$ in the basic solution, we obtain $x_B = B^{-1}b$

So if B is a basis of A, then we obtain the associated basic solution $x = (x_B, x_N)^T$ as

$$x_B = B^{-1}b, \; x_N = 0$$

3.3 Example (continued)

Basis 3

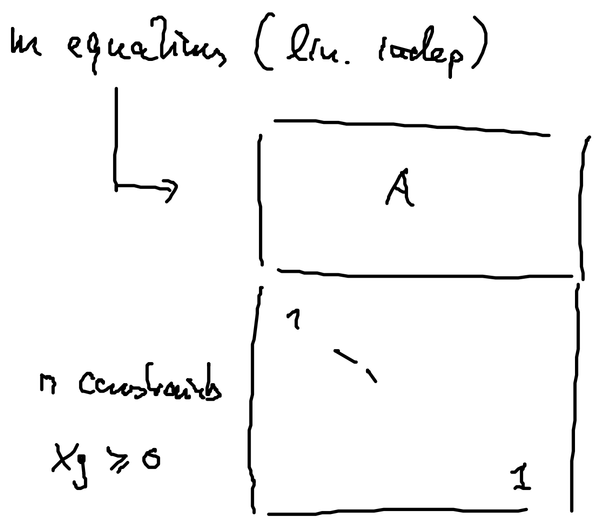
$$B = \begin{pmatrix} 1 & 1 & 1 & \\ & 1 & & \\ & & 1 & \\ 3 & & 1 & 1 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \begin{pmatrix} 1 & -1 & -1 & \\ & 1 & & \\ & & 1 & \\ -3 & 3 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow \quad x_B = \begin{pmatrix} x_2 \\ x_1 \\ x_3 \\ x_7 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 1 & -1 & -1 & \\ & 1 & & \\ & & 1 & \\ -3 & 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

A basic solution x is called a basic feasible solution (bfs for short) if $x \geq 0$, i.e., x is a feasible solution of the LP

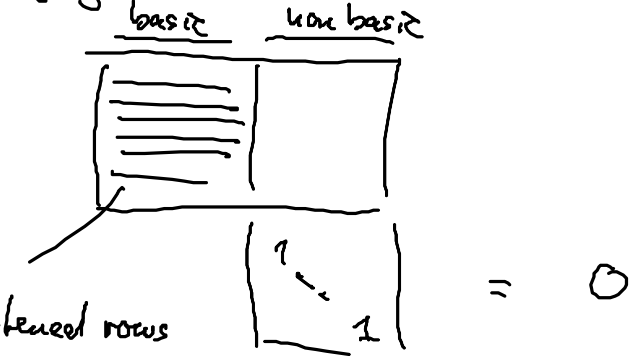
3.3 Example (continued)

The basic solution of basis 1 is feasible, those of basis 2 and basis 3 are not.



If we want n lin. ind. active constraints
we must put $n-m$ of the x_j to 0
call them the non-basic variables
remaining ones the basic variables $B(1), \ldots, B(m)$

the rows must be linearly independent with identity matrix belonging to non-basic variable



these shortened rows must be lin. indep

these are m, row rank = column rank $\Rightarrow$ A_B has m lin. ind. columns

In the above example we went the other way round.

we first choose m lin. ind. columns of A (the basic columns)

set the non-basic variables to 0 and obtain altogether n lin. indep. active constraints $\Rightarrow$ a basic solution

we can calculate the coordinates of the basic solution by row operations transforming A_B to identity matrix or multiplying A_B from the left by B^{-1}