Chapter 2:
The Geometry of Linear Programming

(cp. Bertsimas & Tsitsiklis, Chapter 2)

this week + next week

VL + UE exchange TH – FR

$$\left.\begin{aligned} a_i^T x &\geq b_i \\ &\leq b_i \\ &= b_i \end{aligned}\right\} \text{ constraints}$$

some $x_j \geq 0$, some ≤ 0 some unconstrained

min (max) $c^T x$ objective function

$\Downarrow$ massage to

$Ax \geq b$ $\quad A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$ $m \times n$ matrix

$(x \geq 0)$

$$\left.\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}\right\} \text{ standard form (used by simplex algo)}$$

Polyhedra and Polytopes

Definition 2.1.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

- **a** set $\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ is called polyhedron
- **b** $\{x \mid A \cdot x = b,\ x \geq 0\}$ is polyhedron in standard form representation

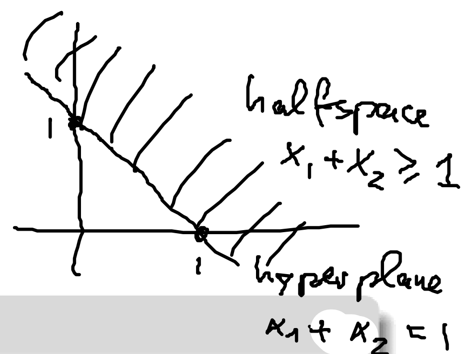
Definition 2.2.

- **a** Set $S \subseteq \mathbb{R}^n$ is bounded if there is $K \in \mathbb{R}$ such that

$$\|x\|_\infty \leq K \qquad \text{for all } x \in S.$$

- **b** A bounded polyhedron is called polytope.

Hyperplanes and Halfspaces



Definition 2.3.

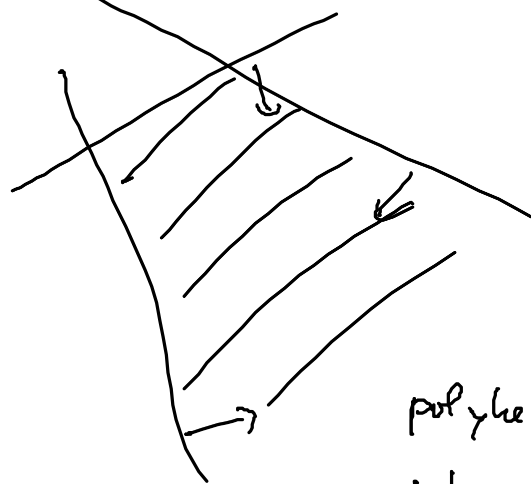
Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$:

- **a** set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is called hyperplane
- **b** set $\{x \in \mathbb{R}^n \mid a^T \cdot x \geq b\}$ is called halfspace

Remarks

- Hyperplanes and halfspaces are convex sets.
- A polyhedron is an intersection of finitely many halfspaces.

$$a_1^T x \geq b_1$$
$$a_2^T x \geq b_2$$
$$\vdots$$
$$a_m^T x \geq b_m$$



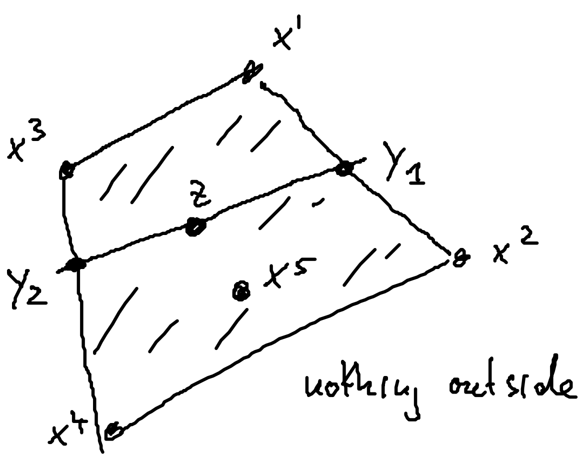
polyhedron as intersection of finitely many halfspaces

Convex Combination and Convex Hull

Definition 2.4.

Let $x^1, \dots, x^k \in \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \dots + \lambda_k = 1$.

- **a** The vector $\sum_{i=1}^{k} \lambda_i \cdot x^i$ is a convex combination of $x^1, \dots, x^k$.
- **b** The convex hull of $x^1, \dots, x^k$ is the set of all convex combinations.



z is a convex comb of x_1 -- x_k

$$y_1 = \lambda_1 x^1 + \lambda_2 x^2 \qquad \lambda_1 + \lambda_2 = 1 \qquad \lambda_i \geq 0$$

$$y_2 = \lambda_3 x^3 + \lambda_4 x^4 \qquad -.$$

$z = \mu_1 \cdot y_1 + \mu_2 y_2 \qquad \mu_i \; - \cdot \cdot$

$z = \underbrace{\mu_1 \lambda_1}_{\bar{\lambda}_1} x^1 + \underbrace{\mu_1 \lambda^2}_{\bar{\lambda}_2} x^2 + \underbrace{\mu_2 \lambda_3}_{\bar{\lambda}_3} x^3 + \underbrace{\mu_2 \lambda^4}_{\bar{\lambda}_4} \cdot x^4 \qquad \geq 0 \qquad \sum = 1$

Convex Sets, Convex Combinations, and Convex Hulls

Theorem 2.5.

- a The intersection of convex sets is convex.
- b Every polyhedron is a convex set.
- c A convex combination of a finite number of elements of a convex set also belongs to that set.
- d The convex hull of finitely many vectors is a convex set.

Proof: ...

a) let X_i, $i \in I$ be convex sets, let $X := \bigcap_{i \in I} X_i$

show that X is convex

i.e. $\forall x, y \in X$, $\forall \lambda \in [0,1]$: $\underbrace{\lambda x + (1-\lambda) x}_{=: z} \in X$

let $x, y \in X \Rightarrow x, y \in X_i \quad \forall i$

$\underset{X_i \text{ conv.}}{\Rightarrow} z \in X_i \quad \forall i$

$\Rightarrow z \in X \quad \square$

b) Polyhedron = Intersection of (finitely many) halfspaces

$\uparrow$ are convex

c) Show by induction that the convex combination of any l points $\in X$ is in X (X is our convex set)

$l = 1$ trivial

$l = 2$ by definition of convexity

$l-1 \to l$ (inductive step)

$\mu_l \neq 0, 1$ otherwise done

Consider $\mu_1 x^1 + \dots \mu_{l-1} x^{l-1} + \mu_l x^l$ (convex comb.)

$$\underbrace{\frac{\mu_1}{1-\mu_l}}_{\nu_1}, \dots, \underbrace{\frac{\mu_{l-1}}{1-\mu_l}}_{\nu_{l-1}} \geq 0 \qquad \text{sum} = 1$$

$\Rightarrow$ $\underbrace{\nu_1 x^1 + \dots + \nu_{l-1} x^{l-1}}_{=: y} \in X$ by inductive assumption

$\Rightarrow$ $\underbrace{(1-\mu_l) y + \mu_l \cdot x^l}$ $\in X$ (as convex comb. of 2 points in X)

$= \mu_1 x^1 + \dots + \mu_l x^l$

d) convex hull $X \overset{\text{Def}}{=}$ set of all convex comb. of $x^1, \dots, x^k$

show that X is convex

Let $x \in X$, $y \in X$, $0 \leq \lambda \leq 1$, show that $\underbrace{\lambda x + (1-\lambda) y}_{=: z} \in X$

$x = \sum_{i=1}^{k} \alpha_i x^i$ $\qquad$ $y = \sum_{i=1}^{k} \beta_i x^i$

convex comb

$$\Rightarrow z = \sum_i \underbrace{[\lambda \alpha_i + (1-\lambda)\beta_i]}_{\geq 0,\ \Sigma = 1} x^i \qquad \square$$

Corollary 2.6.

The convex hull of $x^1, \ldots, x^k \in \mathbb{R}^n$ is the smallest (w.r.t. inclusion) convex subset of $\mathbb{R}^n$ containing $x^1, \ldots, x^k$.

Proof: ...

Let $X :=$ convex hull of $x^1, \ldots, x^k$

$=$ set of all convex comb. of $x^1, \ldots, x^k$

know that X is convex

must show: X is the smallest $\overset{\text{convex}}{\text{set}}$ (under $\subseteq$) containing $x^1, \ldots, x^k$

Let $Y := \bigcap\limits_{\substack{C \subseteq \mathbb{R}^n \\ C \text{ convex} \\ C \text{ contains } x^1, \ldots, x^k}} C$ $\leftarrow$ $\mathbb{R}^n$ and X are among them

Y is convex as intersection of convex sets

is the smallest convex set containing $x^1, \ldots, x^k$

if B would be smaller, i.e. $B \subsetneqq Y$

we get a contradiction, because B takes part in the intersection

$Y \subseteq X$ (X takes part in intersection)

must show that $X \subseteq Y$

$x \in X \underset{\substack{\text{Def} \\ \text{conv. hull}}}{\Rightarrow} x = \sum_{i=1}^{k} \underset{\substack{\uparrow \\ \text{convex comb}}}{\lambda_i x^i}$

Y contains $x^1, \dots, x^k \underset{2.5\,c}{\Rightarrow} Y$ contains x $\square$

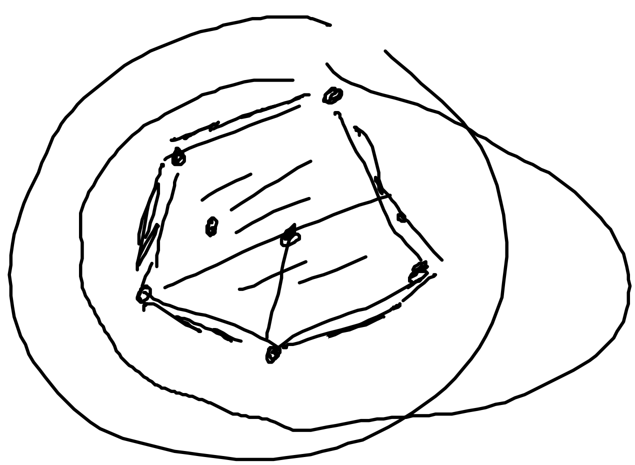
Remark: Have 2 interpretations of convex hull of $x^1, \dots, x^k$

(1) Def (interior description)

take all convex combinations

(2) exterior description

take intersection of all convex sets containing $x^1, \dots x^k$



Extreme Points and Vertices of Polyhedra

Definition 2.7.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. $Ax \geq b$

a $x \in P$ is an extreme point of P if

$$x \neq \lambda \cdot y + (1 - \lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\},\ 0 \leq \lambda \leq 1,$$

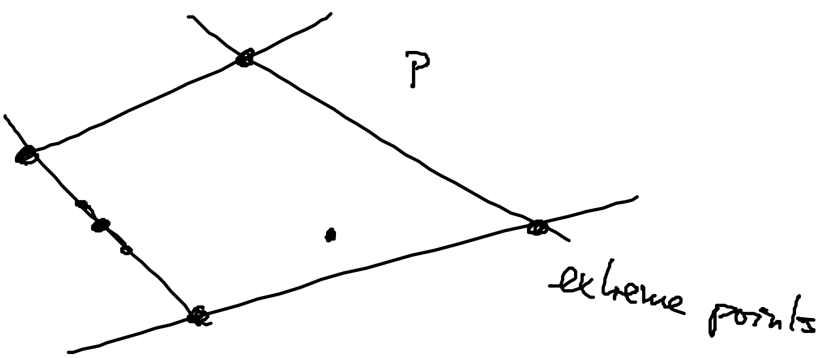
i. e., x is not a convex combination of two other points in P.

b $x \in P$ is a vertex of P if there is some $c \in \mathbb{R}^n$ such that

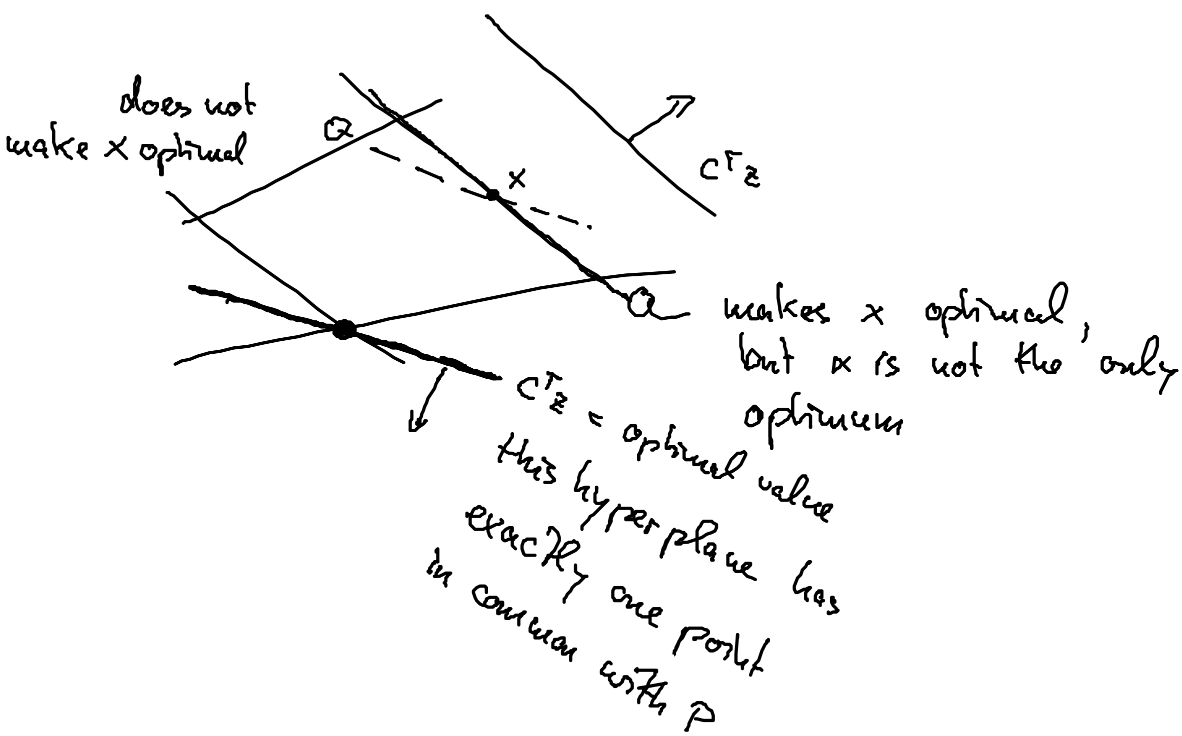
$$c^T \cdot x < c^T \cdot y \qquad \text{for all } y \in P \setminus \{x\},$$

i. e., x is the unique optimal solution to the LP $\min\{c^T \cdot z \mid z \in P\}$.

a)



b)



Active and Binding Constraints

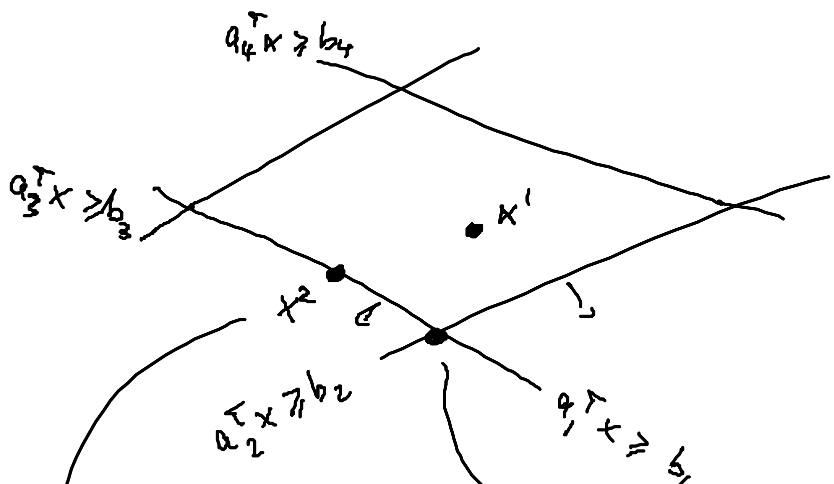
In the following, let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by

$$
\begin{aligned}
{a_i}^T \cdot x &\geq b_i && \text{for } i \in M_1,\\
{a_i}^T \cdot x &= b_i && \text{for } i \in M_2,
\end{aligned}
$$

with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, for all i.

Definition 2.8.

If $x^* \in \mathbb{R}^n$ satisfies ${a_i}^T \cdot x^* = b_i$ for some i, then the corresponding constraint is active (or binding) at x^*.



x_1, x_2 plane in $\mathbb{R}^3$ $\quad x_3 = 0$

x^1 has one active constraint if $x^1_3 = 0$

2 active constraints

3 active constraints

Constraints here

$$
\begin{bmatrix} a_{11} & \cdots & a_{13} \\ a_{21} & \cdots & a_{23} \\ a_{31} & - & a_{43} \\ a_{41} & \cdots & a_{43} \\ \hline 0 & 0 & 1 \end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\begin{matrix} \geq \\ \\ = \end{matrix}
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \hline 0 \end{pmatrix}
$$

also linearly independent rows

Basic Facts from Linear Algebra

Theorem 2.9.

Let $x^* \in \mathbb{R}^n$ and $I = \{i \mid {a_i}^T \cdot x^* = b_i\}$. The following are equivalent:

i. there are n vectors in $\{a_i \mid i \in I\}$ which are linearly independent;
ii. the vectors in $\{a_i \mid i \in I\}$ span $\mathbb{R}^n$;
iii. x^* is the unique solution to the system of equations ${a_i}^T \cdot x = b_i$, $i \in I$.

Vertices, Extreme Points, and Basic Feasible Solutions

Definition 2.10.

a. $x^* \in \mathbb{R}^n$ is a basic solution of P if
   - all equality constraints are active and
   - there are n linearly independent constraints that are active.
b. A basic solution satisfying all constraints is a basic feasible solution.

Theorem 2.11.

For $x^* \in P$, the following are equivalent:

i. x^* is a vertex of P;
ii. x^* is an extreme point of P;
iii. x^* is a basic feasible solution of P.

Proof: ...