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14. Exercise sheet

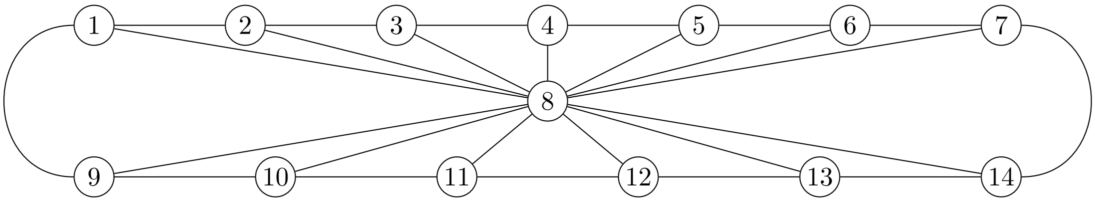
Due-date: Friday, 14.02.2014, *before* the exercise session has started

This is the last exercise sheet.

Exercise 83 **5 bonus points**

A node coloring of an undirected graph $G = (V, E)$ using k colors is a function $f : V \to \{1, \ldots, k\}$, such that $f(u) \neq f(v)$ for each $(u, v) \in E$.

Find such a coloring for the given graph with 3 colors or give a reason why this is not possible.



Exercise 84 **8 bonus points**

Consider the following decision problems:

- Set Basis: Given a set V and a set $\mathcal{C}$ of subsets of V, i.e. $\mathcal{C} \subset 2^V$.

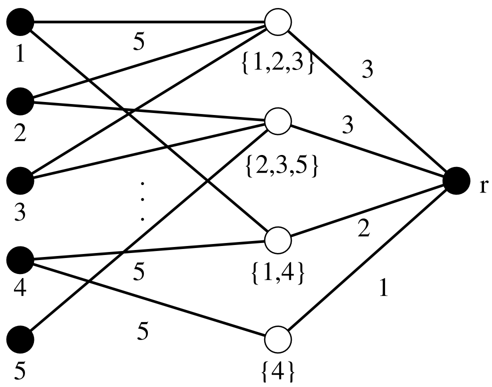
  Is there a set of pairwise disjoint sets C_i in $\mathcal{C}$ such that $\dot{\bigcup}_i C_i = V$?
- Steiner Tree: Given an undirected graph $G = (V, E)$ with non-negative integral costs c_e for all $e \in E$, a subset $T \subset V$, called *terminals*, and an integer k.

  Is there a tree S in G which contains all nodes in T with cost at most k?

Given a Set Basis Problem, we construct an instance of the Steiner Tree Problem as follows.

The set of vertices of G contains V, $\mathcal{C}$ and an additional vertex r. We add edges $\{r, C_i\}$ for all sets $C_i \in \mathcal{C}$, each with a cost of $|C_i|$. Then, for each vertex C_i, we add the edges $\{v, C_i\}$, if v is an element of set C_i, the cost is set to $|V|$. Let $V \cup \{r\}$ be the set of terminals and we choose $k = |V|^2 + |V|$.

For the Set Basis problem, given by set $V = \{1, 2, 3, 4, 5\}$ and $\mathcal{C} = \Big\{\{1, 2, 3\}, \{2, 3, 5\}, \{1, 4\}, \{4\}\Big\}$ our construction would generate the depicted instance of the Steienr Tree Problem. The terminals are the black vertices, $k = 25 + 5 = 30$.



Set Basis is $\mathcal{NP}$-complete. Prove that Steiner Tree is $\mathcal{NP}$-complete as well, using the construction above.

Exercise 85 **7 bonus points**

The Rescource Constrained Shortest Path Problem(RCSP) is similar to the usual shortest path problem. Given a directed graph $D = (V, A)$ with costs on edges $c : A \to \mathbb{R}^+$, two nodes $s, t \in V$, another resource function $r \to \mathbb{R}^+$ and a resource bound R.

We want to find the shortest s-t path which does not exceed the resource bound B, i.e., $\sum_{e \in s-t \text{ - path}} r(e) \leq B$.

(a) Prove that the decision problem of RCSP is weakly $\mathcal{NP}$-complete.

(b) Describe a pseudopolynomial algorithm for RCSP.

(c) Consider the same problem with k resource functions $r_i : A \to \mathbb{R}^+$ and k resource bounds B_i. Is the problem still weakly $\mathcal{NP}$-complete? How could you adopt your algorithm to this case?

Exercise 86 **Tutorial session – 0 points**

An Internet Service Provider (ISP) stores all the IP adresses its customers are accessing every minute. We assume that each customer does not acceess more than one IP per minute. Recently, the provider's server were the target of a hacking attack. During t successive minutes t different IP adresses $i_1, \ldots, i_t$ have been accessed, but strangley, no single customer has accessed those t addresses in the t minutes. The provider asks himself whether there is a group of k customers, who could been used to execute the attack.

Prove via the Node Cover Problem, that this problem is $\mathcal{NP}$-complete. You can assume that we know, that Node Cover is $\mathcal{NP}$-complete.

Exercise 87 **Tutorial session – 0 points**

We repeat some problem definitions:

TSP Decision Problem:

Given: Undirected graph $G = (V, E)$ with edge cost $c : E \to \mathbb{N}_0$ and an integer $K \in \mathbb{N}_0$.

Question: Is there a simple cycle in G, visiting all vertices ("Tour") with length $\leq K$ (length = sum of all edges along the cycle)?

Hamilton Circuit:

Given: Undirected graph G.

Question: Is there a Hamiltonian Cycle in G, i.e. a simple cycle visiting all vertices in G?

Hamiltonian Path:

Given: Undirected graph G.

Question: Is there Hamiltonian Path in G, i.e. a simple path visiting all vertices in G?

u-v-Hamiltonian Path:

Given: Undirected graph $G = (V, E)$ and two distinct vertices $u, v \in V$.

Question: IS there a Hamiltonian Path in G, starting at u and ending at v?

Comment: All problems can be considered on directed graphs. The goal is to find an directed cycle rsp. path visiting all vertices.

We assume that it is shown, that the problem Hamilton Circuit with an undirected graph is $\mathcal{NP}$-complete.

Prove the $\mathcal{NP}$-completeness of the following problems:

(a) TSP

(b) u-v-Hamiltonian Path

(c) Hamiltonian Path

(d) all problems on directed graphs