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8. Exercise sheet

Due-date: Friday, 13.12.2013, *before* the exercise session has started

This is the first exercise sheet in the second half of the semester.

Exercise 43 **4 points**

Prove that in every undirected graph $G = (V, E)$ there is a cut of size at least $\frac{|E|}{2}$.
Hint: You can use a constructive proof by giving a deterministic or even a randomized algorithm.

Exercise 44 **3+4 points**

Let $G = (V, E)$ be a directed graph and $c : E \to \mathbb{R}$ a cost function for the edges.

(a) Prove Lemma 5.14 presented in the lecture:
If c is integer-valued, $C := 2\max_{a\in A}|c_a|+1$, and there is no negative-cost dicircuit, then Ford's Algorithm terminates after at most $C \cdot n^2$ iterations.

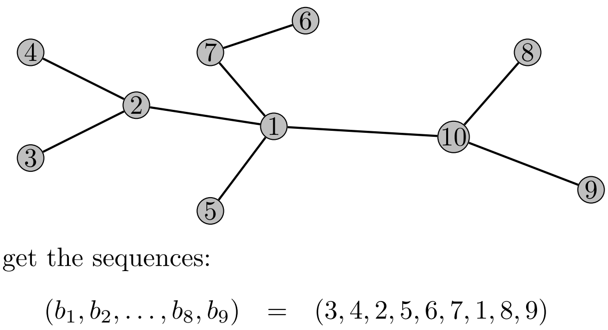
(b) Assume there is no negative-cost dicircuit. Provide an example for which Ford's Algorithm can have a running time exponential in $n = |V|$ and $m = |E|$. For that purpose you have to describe a family of graphs, a function c and a sequence in which the edges are considered by the algorithm. A family of graphs means, you have to find a graph with at least n vertices for every given $n \in \mathbb{N}$. *Hint: You may use parallel edges in the graphs.*

Exercise 45 **2+5+2 points**

Let $n \in \mathbb{N}$ with $n \geq 2$ and let $\mathcal{T}(n)$ be the set of spanning trees of the complete undirected graph with n nodes. We want to prove Cayley's theorem stating that the number of spanning trees of such a graph is $t(n) = |\mathcal{T}(n)| = n^{n-2}$.

(a) Prove that a tree with maximum node-degree $d_{\max}$ has at least $d_{\max}$ many vertices of degree 1 (*leaves*).

(b) Let T be a tree with vertices $\{1, \dots, n\}$ and set $T_1 := T$. Let b_1 be a leaf of T having the smallest number among all leaves. Let a_1 be the unique neighbor of b_1. We delete b_1 and the edge $\{a_1, b_1\}$ and obtain the tree T_2 (why is T_2 indeed a tree?). Recursively repeating the procedure we get a sequence of vertices $(a_1, a_2, \dots, a_{n-1})$.



For the example we get the sequences:

$$\begin{aligned}(b_1, b_2, \dots, b_8, b_9) &= (3, 4, 2, 5, 6, 7, 1, 8, 9)\\(a_1, a_2, \dots, a_8, a_9) &= (2, 2, 1, 1, 7, 1, 10, 10, 10)\end{aligned}$$

We define:

$$f : \mathcal{T}(n) \to \{1, \dots, n\}^{n-2}$$

$$T \mapsto (a_1, a_2, \ldots, a_{n-2})$$

f computes the so-called *Prüfer-code*. The *Prüfercode* for the depicted tree is $(2, 2, 1, 1, 7, 1, 10, 10)$.

Prove that f is a bijection by specifying the inverse mapping of f.

(c) Prove that $t(n) = n^{n-2}$.

Exercise 46 **Tutorial session – 0 points**

A topological ordering on a connected directed graph $D = (V, E)$ is given by a labeling $\ell : V \to \mathbb{N}$ of the vertices such that for $a = (v, w) \in A$ it holds that $\ell(v) < \ell(w)$. (This implies that D is acyclic.) Let D be a connected directed acyclic graph.

(a) Prove that in D there is a vertex with $|\delta^-(v)| = 0$.

(b) Based on (a) give an algorithm (in pseudocode) that constructs a topological ordering of D in linear time.

(c) Find an algorithm based on DFS that constructs a topological ordering.

Exercise 47 **Tutorial session – 0 points**

Formulate the following statement in terms of a graph and prove or disprove it: In a group of six persons there are always three persons who know or do not know each other. (Understand knowing each other as a symmetrical relation.)

Exercise 48 **Tutorial session – 0 points**

Consider a digraph $D = (V, A)$ with conservative arc costs $c : A \to \mathbb{R}$. Let $s, t \in V$ such that t is reachable from s. Prove that the least cost of an s-t-path in D equals the maximum of $\pi(t) - \pi(s)$ where π is a feasible potential of (D, c).