Approximability of MAX 2SAT

In Chapter 5 we stated the following hardness result:

Theorem 9.6.

Unless $P = NP$, there is no $(7/8 + \varepsilon)$-approximation algorithm for MAX E 3SAT for any constant $\varepsilon > 0$.

Proof: Via PCP Theorem (see Williamson & Shmoys, Chapter 16.3). □

Theorem 9.7.

Unless $P = NP$, there is no α-approximation algorithm for MAX 2SAT for $\alpha > \frac{433}{440} \approx 0.984$.

Proof:. . . □

Approximation-Preserving Reductions

Consider two optimization problems $\mathcal{P}$ and $\mathcal{P}'$ with corresponding sets of instances $X_{\mathcal{P}}$ and $X_{\mathcal{P}'}$, respectively.

Definition 9.8 (L-Reduction).

An L-reduction from $\mathcal{P}$ to $\mathcal{P}'$ with parameters $a, b > 0$ is a map $f : X_{\mathcal{P}} \to X_{\mathcal{P}'}$ such that for all $I \in X_{\mathcal{P}}$:

i. $I' := f(I)$ can be computed in time polynomial in the size of I;
ii. $\mathrm{OPT}(I') \leq a \cdot \mathrm{OPT}(I)$;
iii. given a solution of value V' to I', one can compute in polynomial time a solution of value V to I such that

$$|\mathrm{OPT}(I) - V| \leq b \cdot |\mathrm{OPT}(I') - V'| \ .$$

Example: The reduction from MAX E 3SAT to MAX 2SAT in the proof of Theorem 9.7 is an L-reduction with parameters $a = \frac{55}{7}$ and $b = 1$.

Approximation-Preserving Reductions

Theorem 9.9.

For maximization problems $\mathcal{P}$ and $\mathcal{P}'$, if there is an L-reduction from $\mathcal{P}$ to $\mathcal{P}'$, and there is an α-approximation algorithm for $\mathcal{P}'$, then there is an $\big(1 - ab(1-\alpha)\big)$-approximation algorithm for $\mathcal{P}$.

Proof:... □

Theorem 9.10.

For minimization problems $\mathcal{P}$ and $\mathcal{P}'$, if there is an L-reduction from $\mathcal{P}$ to $\mathcal{P}'$, and there is an α-approximation algorithm for $\mathcal{P}'$, then there is an $\big(ab(\alpha-1)+1\big)$-approximation algorithm for $\mathcal{P}$. □

Examples of L-Reductions

Lemma 9.11.

There is an L-reduction with parameters $a = \frac{27}{7}$ and $b = 1$ from MAX E3SAT to MAX 2SAT.

Proof:... □

Corollary 9.12.

Unless $P = NP$, there is no α-approximation algorithm for MAX 2SAT for $\alpha > \frac{209}{216} \approx 0.968$.

Lemma 9.13.

There is an L-reduction with parameters $a = 2\Delta$ and $b = 1$ from Vertex Cover in bounded degree graphs to the Steiner Tree Problem.

Proof:... □