Chapter 9:
The Hardness of Approximation

(cp. Williamson & Shmoys, Chapter 16)

MaxClique Problem

Given: Undirected graph $G = (V, E)$.

Task: Find $V' \subseteq V$ maximizing $|V'|$ with all nodes in V' pairwise adjacent.

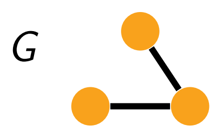
Notation: The size of a largest clique $V' \subseteq V$ in G is denoted by $\omega(G)$.

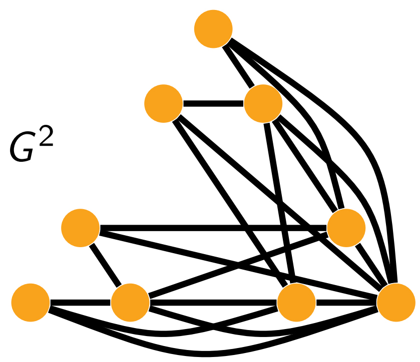
Definition 9.1 (product graph).

For an undirected graph $G = (V, E)$ let $G^k = (V^k, E_k)$ where V^k is the set of all k-tuples of nodes in V and E_k is defined by

$$E_k := \{(u_1, \ldots, u_k)(v_1, \ldots, v_k) \mid u_i = v_i \text{ or } u_i v_i \in E \text{ for all } i\} \ .$$

Example:





A Constant-Factor Approximation for MaxClique?

Lemma 9.2.

$$\omega(G^k) = \omega(G)^k$$

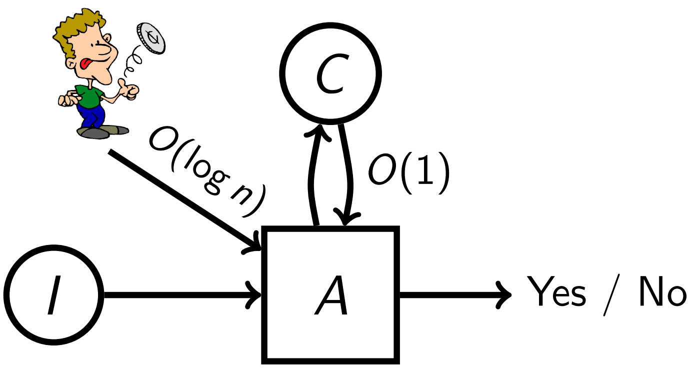
Proof:. . . □

Observation 9.3.

If there is an α-approximation algorithm for MaxClique for some fixed $\alpha < 1$, then there is a polynomial-time approximation scheme.

Proof:. . . □

Another Characterization of NP



correct answer is "Yes" $\implies$ $\exists$ certificate C: Pr(A outputs "Yes")$= 1$.

correct answer is "No" $\implies$ $\forall$ certificates C: Pr(A outputs "Yes")$< \frac{1}{2}$.

PCP Theorem (Arora, Lund, Motwani, Sudan, and Szegedy 1992)

Every decision problem in NP has a probabilistically checkable proof of constant query complexity and logarithmic randomness complexity.

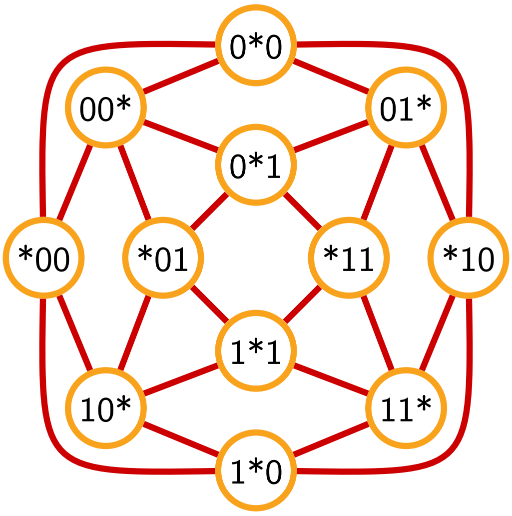
Hardness of Approximation

Theorem 9.4 (Feige, Goldwasser, Lovász, Safra, Szegedy 1991).

There is no 2-approximation algorithm for MaxClique, unless P=NP.

Proof: Form graph G representing PCP system for SAT instance such that

- $\omega(G) = K$ if correct answer is "Yes"
- $\omega(G) < \frac{K}{2}$ if correct answer is "No" □



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Theorem 9.5 (Zuckerman 2007).

There is no $n^{1-\varepsilon}$-approximation algorithm for MaxClique, for any $\varepsilon > 0$, unless $P = NP$.

Bottomline: MaxClique is one of the hardest problems to approximate!