The Multicut Problem

Given: Graph $G = (V, E)$ with edge costs $c_e \geq 0$, $e \in E$; k source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k) \in V \times V$.

Task: Find $F \subseteq E$ of minimum cost $c(F)$ whose removal disconnects s_i from t_i, $i = 1, \ldots, k$.

IP formulation: Let $\mathcal{P}_i$ denote the set of all s_i-t_i-paths

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & \sum_{e \in P} x_e \geq 1 && \text{for all } P \in \mathcal{P}_i,\ i = 1, \ldots, k, \\ & x_e \in \{0, 1\} && \text{for all } e \in E. \end{aligned}$$

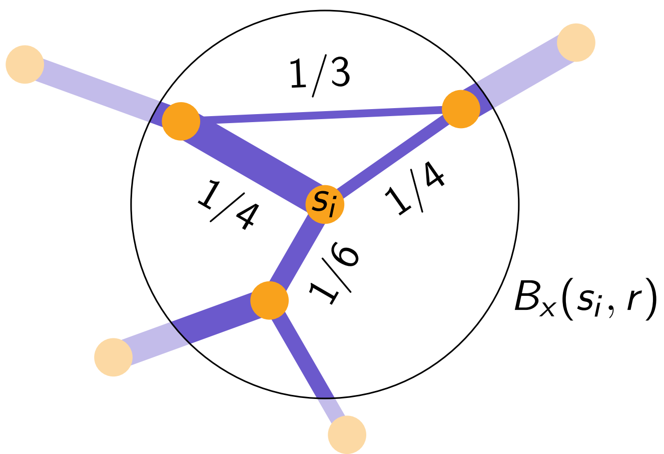
LP relaxation: replace $x_e \in \{0, 1\}$ with $x_e \geq 0$ for all $e \in E$.

- Separation problem consists of k shortest path problems.
- Thus, LP relaxation can be solved in polynomial time.

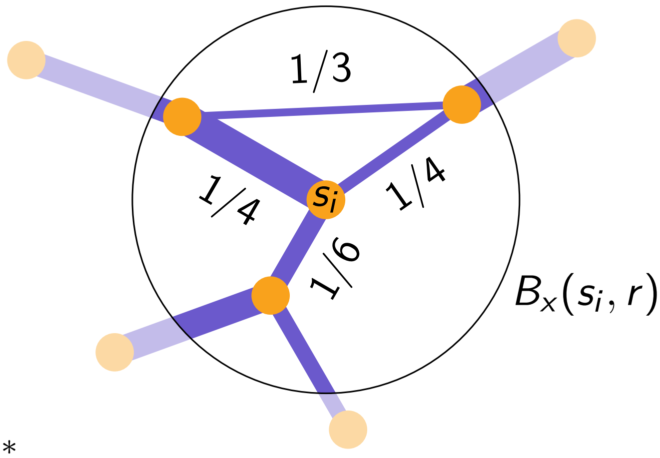
Interpretation of LP Solution

Let x be an optimal LP solution.

- Interpret x_e as the length of edge $e \in E$.
- Let $d_x(u, v) :=$ length of shortest u-v-path w.r.t. edge lengths x_e.
- Notice that $d_x(s_i, t_i) \geq 1$ for all $i = 1, \ldots, k$.
- Let $B_x(s_i, r) := \{v \in V \mid d_x(s_i, v) \leq r\}$ (ball of radius r around s_i).
- Interpret $c_e \cdot x_e$ as the volume of edge $e \in E$.
- LP yields minimum total volume V^* with $d_x(s_i, t_i) \geq 1$, $i = 1, \ldots, k$.



Cut Capacity vs. Volume



$$V_x(s_i, r) := \frac{V^*}{k} + \sum_{\substack{uv \in E: \\ u,v \in B_x(s_i,r)}} c_{uv} \cdot x_{uv} + \sum_{\substack{uv \in E: \\ u \in B_x(s_i,r) \\ v \notin B_x(s_i,r)}} c_{uv} \cdot (r - d_x(s_i, u))$$

Lemma 8.4.

For feasible LP sol. x and s_i, one can efficiently find radius $r < 1/2$ with

$$c\big(\delta(B_x(s_i, r))\big) \leq \big(2\ln(k+1)\big) \cdot V_x(s_i, r) \ .$$

Proof:. . . □

LP Rounding Algorithm

1. compute optimal solution x to LP relaxation; set $F := \emptyset$;
2. for $i = 1, \ldots, k$, if s_i and t_i are connected in $(V, E \setminus F)$, then
3. choose radius r around s_i as in Lemma 8.4;
4. set $F := F \cup \delta\big(B_x(s_i, r)\big)$;
5. set $V := V \setminus B_x(s_i, r)$ and $E := E \setminus \delta\big(B_x(s_i, r)\big)$;

Theorem 8.5.

This is a $\big(4\ln(k+1)\big)$-approximation algorithm for the multicut problem.

Proof:. . . □

Theorem 8.6.

Assuming the *Unique Games Conjecture*, there is no constant-factor approximation algorithm for the multicut problem, unless $P = NP$. □