Chapter 8:
Cuts and Metrics

(cp. Williamson & Shmoys, Chapter 8)

The Multiway Cut Problem

Given: Graph $G = (V, E)$; costs $c_e \geq 0$, $e \in E$; k terminals $s_1, \ldots, s_k \in V$.

Task: Find $F \subseteq E$ of minimum cost $c(F)$ disconnecting all s_i, $i = 1, \ldots, k$.

Simple Algorithm

1. for $i = 1, \ldots, k$ find min-cost $F_i \subseteq E$ isolating s_i from all s_j, $j \neq i$;
2. return $F := \bigcup_{i=1}^{k} F_i$;

Theorem 8.1.
The algorithm is a 2-approximation algorithm for multiway cuts.

Proof:. . . □

Corollary 8.2.
Taking only the cheapest $k - 1$ cuts F_i yields a $(2 - \frac{2}{k})$-approximation algorithm.

Proof:. . . □

IP Formulation of Multiway Cut Problem

In an optimal solution F, subgraph $(V, E \setminus F)$ has exactly k connected components C_i, with $s_i \in C_i$, $i = 1, \dots, k$.

For $u \in V$ and $e \in E$ introduce decision variables

$$x_u^i = \begin{cases} 1 & \text{if } u \in C_i, \\ 0 & \text{otherwise,} \end{cases} \qquad z_e^i = \begin{cases} 1 & \text{if } e \in \delta(C_i), \\ 0 & \text{otherwise.} \end{cases}$$

IP formulation:

$$\begin{aligned} \min \quad & \frac{1}{2} \sum_{e \in E} c_e \sum_{i=1}^{k} z_e^i \\ \text{s.t.} \quad & \sum_{i=1}^{k} x_u^i = 1 && \text{for all } u \in V, \\ & z_e^i \geq |x_u^i - x_v^i| && \text{for all } e = uv \in E, \\ & x_{s_i}^i = 1 && \text{for all } i = 1, \dots, k, \\ & x_u^i \in \{0, 1\} && \text{for all } u \in V,\ i = 1, \dots, k. \end{aligned}$$

LP Relaxation

In an optimal IP solution $z_e^i = |x_u^i - x_v^i|$ and the objective function value is

$$\frac{1}{2} \sum_{e = uv \in E} c_e \cdot \|x_u - x_v\|_1$$

with $x_u \in \Delta_k := \left\{ x \in \mathbb{R}^k_{\geq 0} \mid \sum_{i=1}^{k} x^i = 1 \right\}$ (k-simplex).

LP relaxation:

$$\begin{aligned} \min \quad & \frac{1}{2} \sum_{e = uv \in E} c_e \cdot \|x_u - x_v\|_1 \\ \text{s.t.} \quad & x_{s_i} = e_i && \text{for all } i = 1, \dots, k, \\ & x_u \in \Delta_k && \text{for all } u \in V. \end{aligned}$$

LP Rounding Algorithm

For $i = 1, \dots, k$ and $0 \le r < 1$ let

$$B(e_i, r) := \left\{ u \in V \;\middle|\; \tfrac{1}{2}\|e_i - x_u\|_1 \le r \right\} \; .$$

LP Rounding Algorithm

1. compute optimal solution to LP relaxation;
2. set $C_i := \emptyset$ for all $i = 1, \dots, k$; set $X := \emptyset$; (nodes assigned already)
3. draw $r \in (0, 1)$ uniformly at random;
4. pick random permutation π of $\{1, \dots, k\}$;
5. for $i = 1, \dots, k-1$ set $C_{\pi(i)} := B(e_{\pi(i)}, r) \setminus X$ and $X := X \cup C_{\pi(i)}$;
6. set $C_{\pi(k)} := V \setminus X$ and return $F := \bigcup_{i=1}^{k} \delta(C_i)$;

Theorem 8.3.

The algorithm is a randomized 3/2-approximation algorithm.

Proof:. . . □