Minimum Knapsack Problem

Given: n items with value $v_i \geq 0$ and size $s_i \geq 0$, $i = 1, \ldots, n$; demand D.

Task: Find $X \subseteq \{1, \ldots, n\}$ minimizing $\sum_{i \in X} s_i$ subject to $\sum_{i \in X} v_i \geq D$.

IP formulation:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} s_i \cdot x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} v_i \cdot x_i \geq D \\ & x_i \in \{0,1\} && \text{for all } i = 1, \ldots, n. \end{aligned}$$

Remarks.

- Replace $x_i \in \{0,1\}$ with $0 \leq x_i \leq 1$ to obtain LP relaxation.
- The integrality gap of the LP relaxation is unbounded!
  Bad instance: Let $n = 2$, $v_1 = D - 1$, $v_2 = D$, $s_1 = 0$, $s_2 = 1$.

Stronger LP Relaxation

Notation: Let $[n] := \{1, \ldots, n\}$;
for $A \subseteq [n]$ and $i \in [n] \setminus A$ let $v_i^A := \min\{v_i, D - v(A)\}$.

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} s_i \cdot x_i \\ \text{s.t.} \quad & \sum_{i \in [n] \setminus A} v_i^A \cdot x_i \geq D - v(A) && \text{for all } A \subseteq [n], \\ & x_i \geq 0 && \text{for all } i \in [n]. \end{aligned}$$

Dual LP:

$$\begin{aligned} \max \quad & \sum_{A \subseteq [n]} \big(D - v(A)\big) \cdot y_A \\ \text{s.t.} \quad & \sum_{A \subseteq [n] :\, i \notin A} v_i^A \cdot y_A \leq s_i && \text{for all } i \in [n], \\ & y_A \geq 0 && \text{for all } A \subseteq [n]. \end{aligned}$$

Primal-Dual Algorithm for Minimum Knapsack Problem

1 set $y :\equiv 0$ and $X := \emptyset$;
2 while $v(X) < D$
3 increase y_X until for some $i \notin X$, $\sum_{A \subseteq [n]:\, i \notin A} v_i^A \cdot y_A = s_i$;
4 set $X := X \cup \{i\}$;

Theorem 7.8.
This is a 2-approximation algorithm for the Minimum Knapsack Problem.

Proof:. . . □

Corollary 7.9.
The integrality gap of the strengthened LP relaxation is at most 2. □

Uncapacitated Facility Location Problem

Given: Set of facilities F with opening costs $f_i \geq 0$, $i \in F$;
set of clients D with metric connection costs $c_{ij} \geq 0$, $i \in F$, $j \in D$.

Task: Choose $F' \subseteq F$ and assign each client to nearest facility in F'.

Objective: Minimize $\sum_{i \in F'} f_i + \sum_{j \in D} \min_{i \in F'} c_{ij}$.

IP formulation:

$$
\begin{aligned}
\min \quad & \sum_{i \in F} f_i \cdot y_i + \sum_{i \in F, j \in D} c_{ij} \cdot x_{ij} & \\
\text{s.t.} \quad & \sum_{i \in F} x_{ij} = 1 & \text{for all } j \in D, \\
& y_i - x_{ij} \geq 0 & \text{for all } i \in F, j \in D, \\
& x_{ij}, y_i \in \{0, 1\} & \text{for all } i \in F, j \in D.
\end{aligned}
$$

LP Relaxation and Dual LP

primal LP:

$$\begin{aligned}
\min \quad & \sum_{i\in F} f_i \cdot y_i + \sum_{i\in F, j\in D} c_{ij}\cdot x_{ij} & \\
\text{s.t.} \quad & \sum_{i\in F} x_{ij} = 1 & \text{for all } j\in D,\\
& y_i - x_{ij} \geq 0 & \text{for all } i\in F, j\in D,\\
& x_{ij}, y_i \geq 0 & \text{for all } i\in F, j\in D.
\end{aligned}$$

dual LP:

$$\begin{aligned}
\max \quad & \sum_{j\in D} v_j & \\
\text{s.t.} \quad & \sum_{j\in D} w_{ij} \leq f_i & \text{for all } i\in F,\\
& v_j - w_{ij} \leq c_{ij} & \text{for all } i\in F, j\in D,\\
& w_{ij} \geq 0 & \text{for all } i\in F, j\in D.
\end{aligned}$$

Interpretation of dual LP:

- v_j is total amount that client j wants to pay for being served.
- client j might contribute w_{ij} to facility i for being connected to i.

Primal-Dual Algorithm for Uncapacitated Facility Location

Notation: For the current feasible dual solution (v, w) and $j \in D$ let

$$N(j) := \{i \in F \mid v_j \geq c_{ij}\} \ .$$

1 set $S := D$, $F'' := \emptyset$, $v_j := 0$, $w_{ij} := 0$ for all $i \in F$, $j \in D$;

2 while $S \neq \emptyset$

3 for all $j \in S$ increase v_j and w_{ij} for all $i \in N(j)$ uniformly
until $N(j) \cap F'' \neq \emptyset$ for some $j \in S$ or $\sum_{j\in D} w_{ij} = f_i$ for some $i \notin F''$;

4 if $\sum_{j\in D} w_{ij} = f_i$ for some $i \notin F''$, then $F'' := F'' \cup \{i\}$;

5 if $N(j) \cap F'' \neq \emptyset$ for some $j \in S$, then $S := S \setminus \{j\}$;

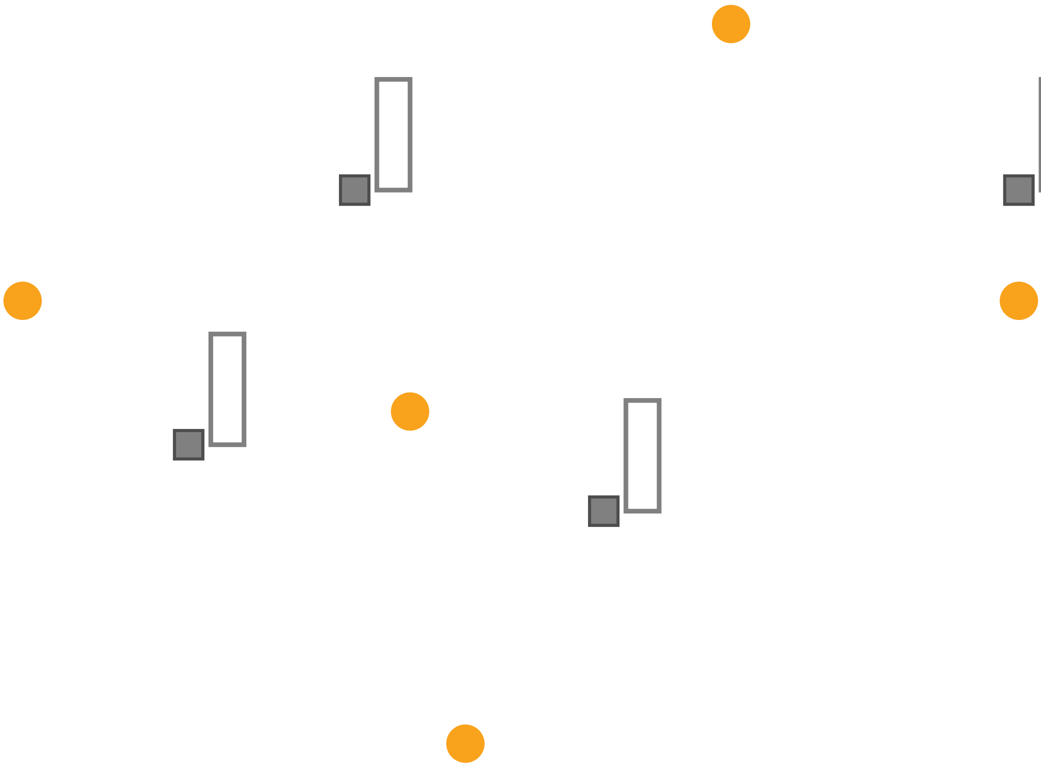
6 set $F' := \emptyset$

7 while $F'' \neq \emptyset$ pick $i \in F''$;

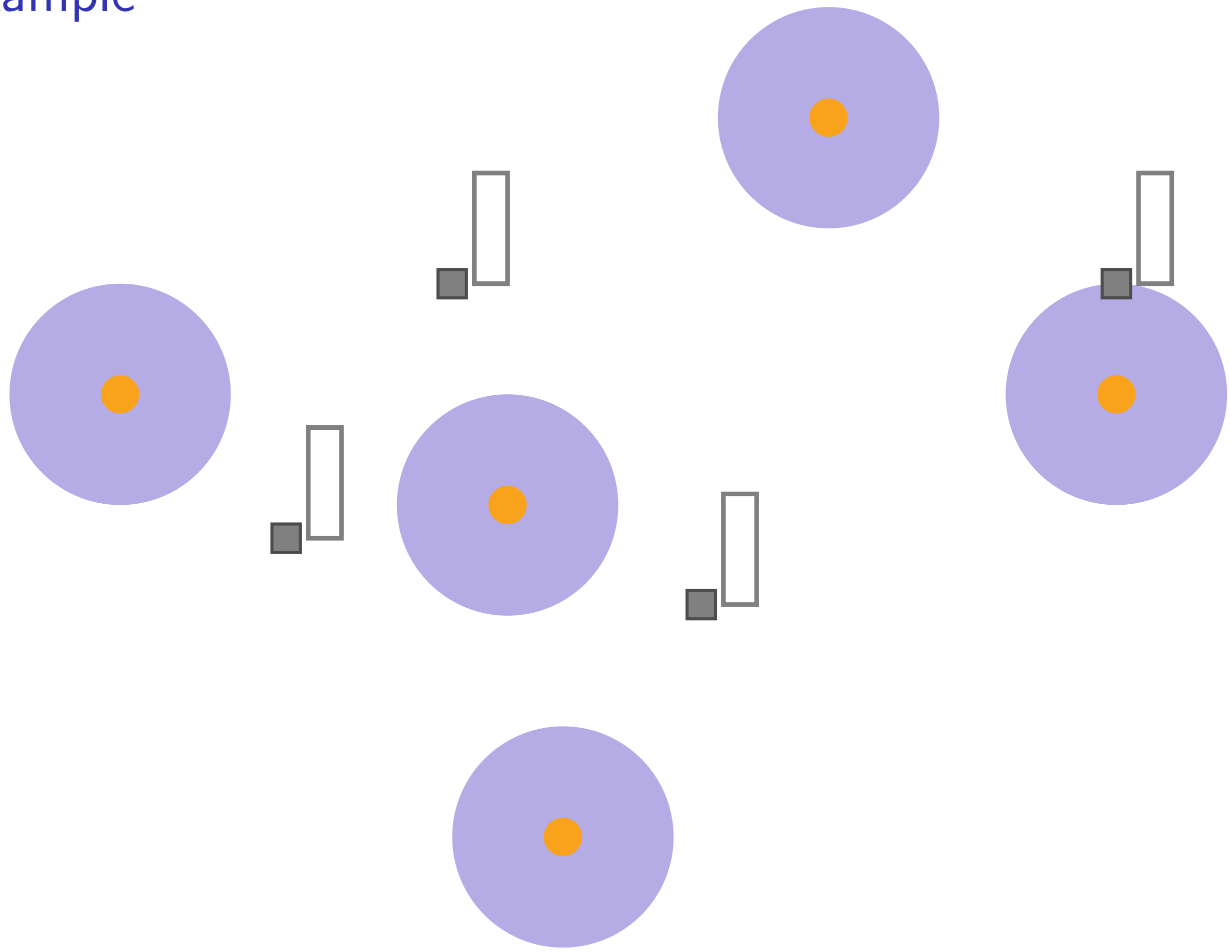
8 $F' := F' \cup \{i\}$; $F'' := F'' \setminus \{h \mid \exists j \in D,\ w_{ij} > 0,\ w_{hj} > 0\}$;

9 open all facilities in F'; connect each $j \in D$ to nearest $i \in F'$;

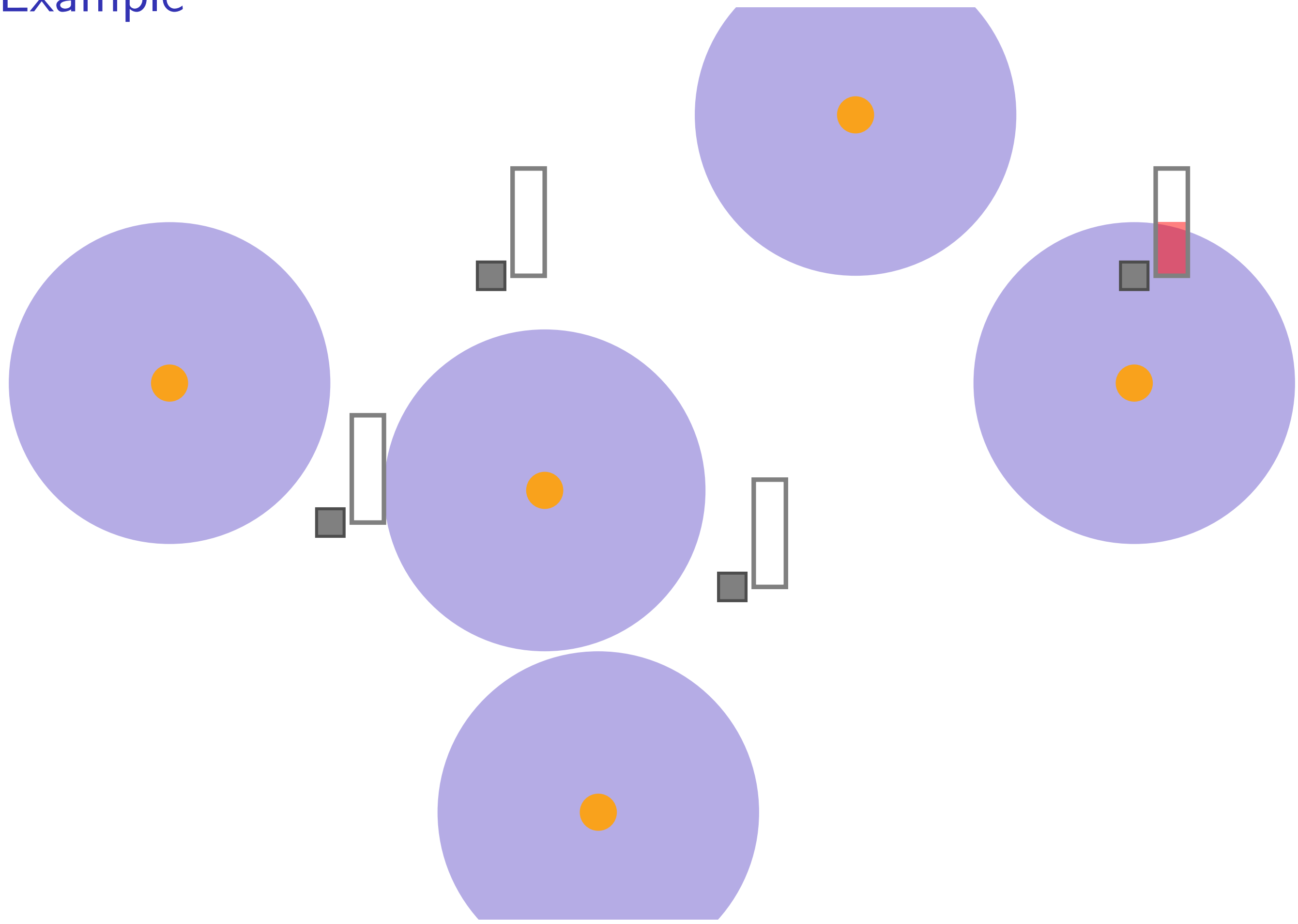
Example



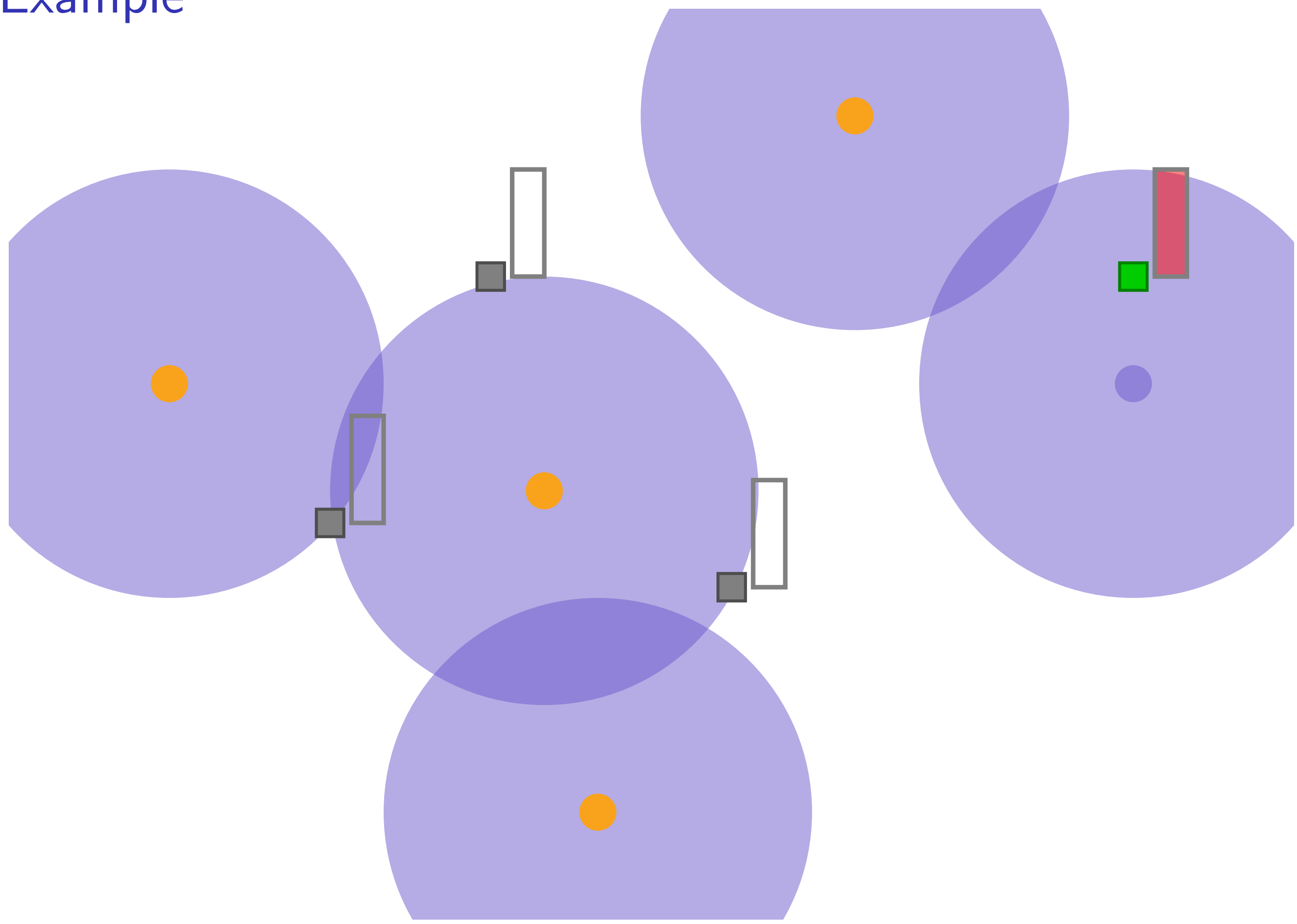
Example



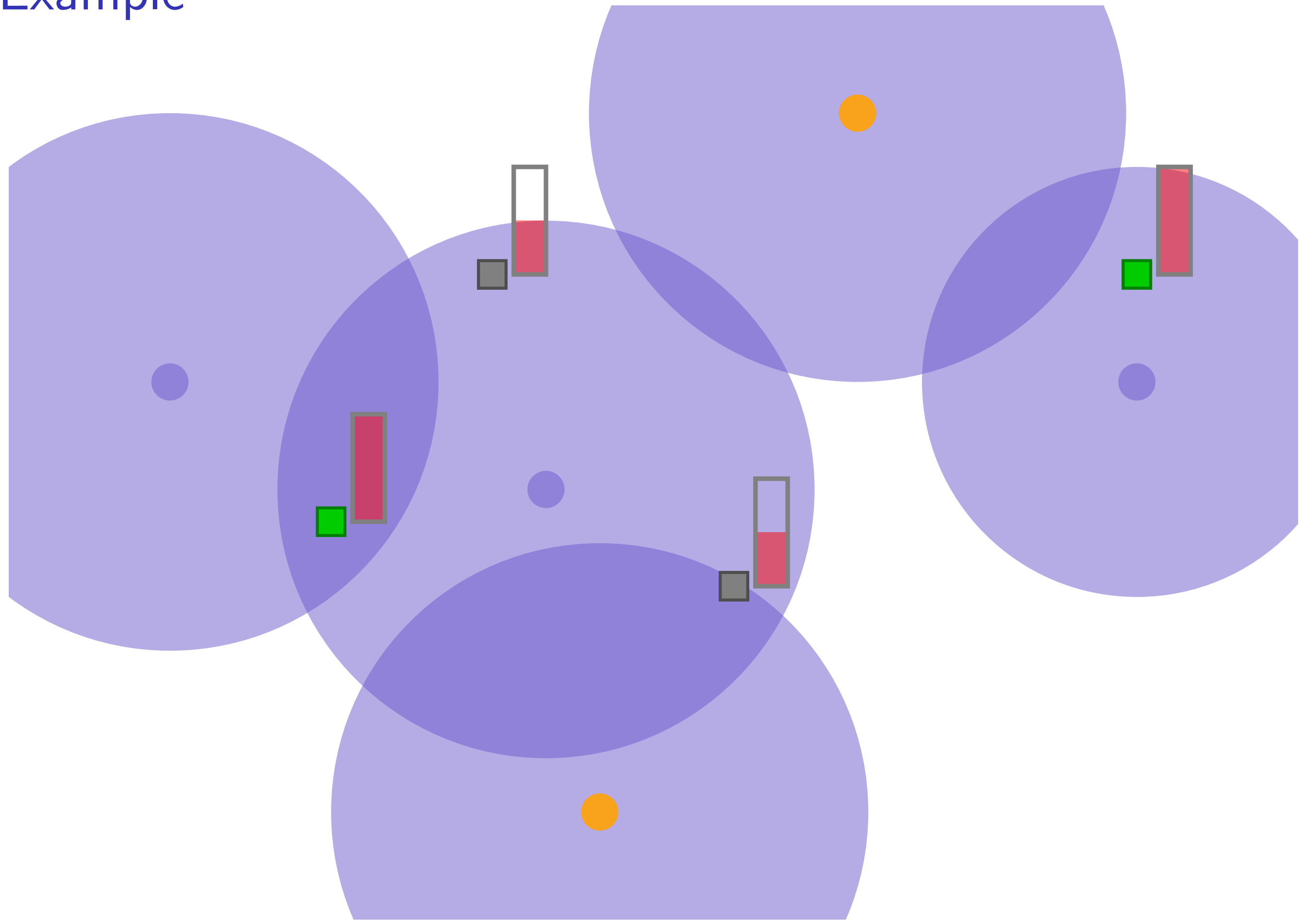
Example



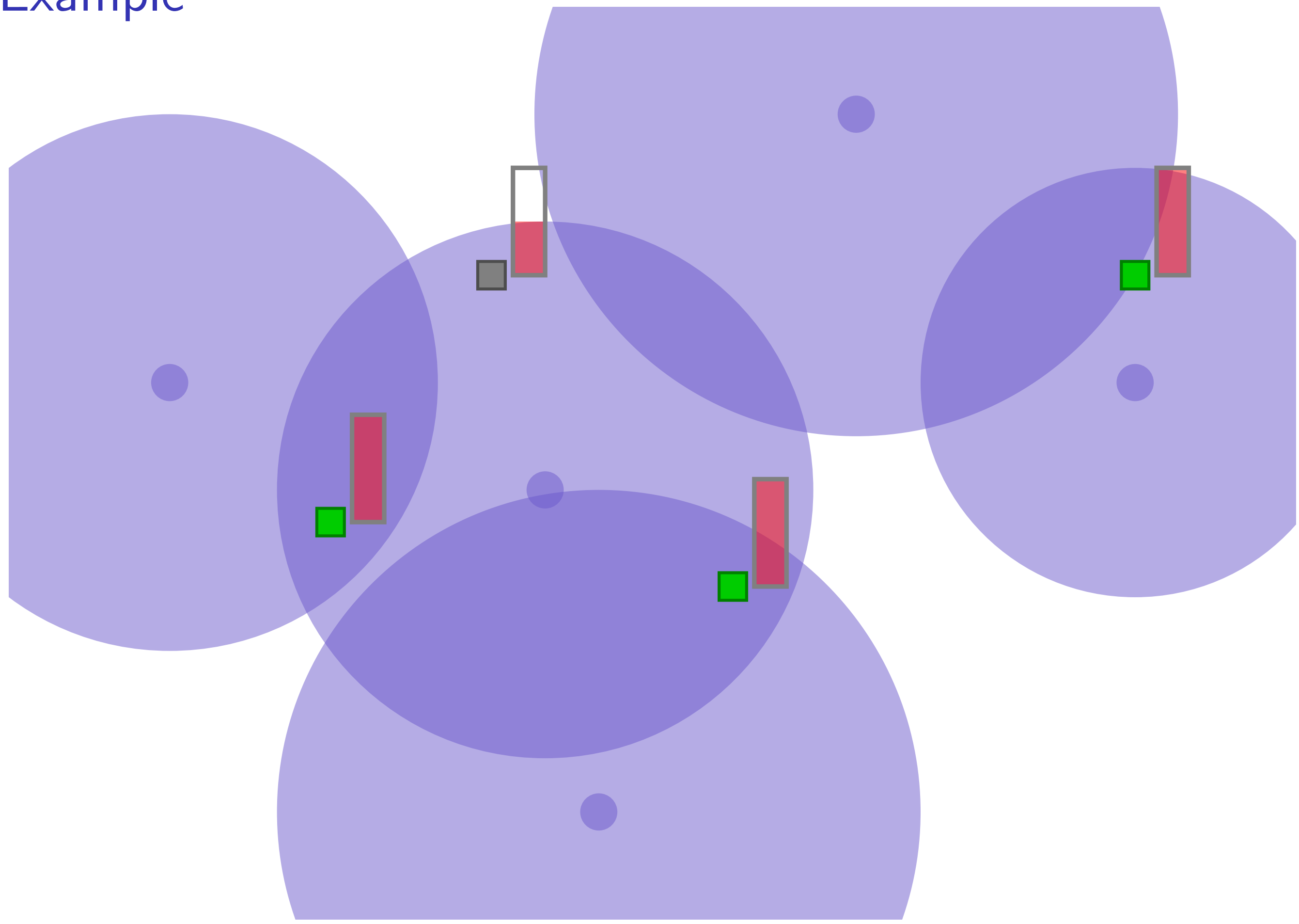
Example



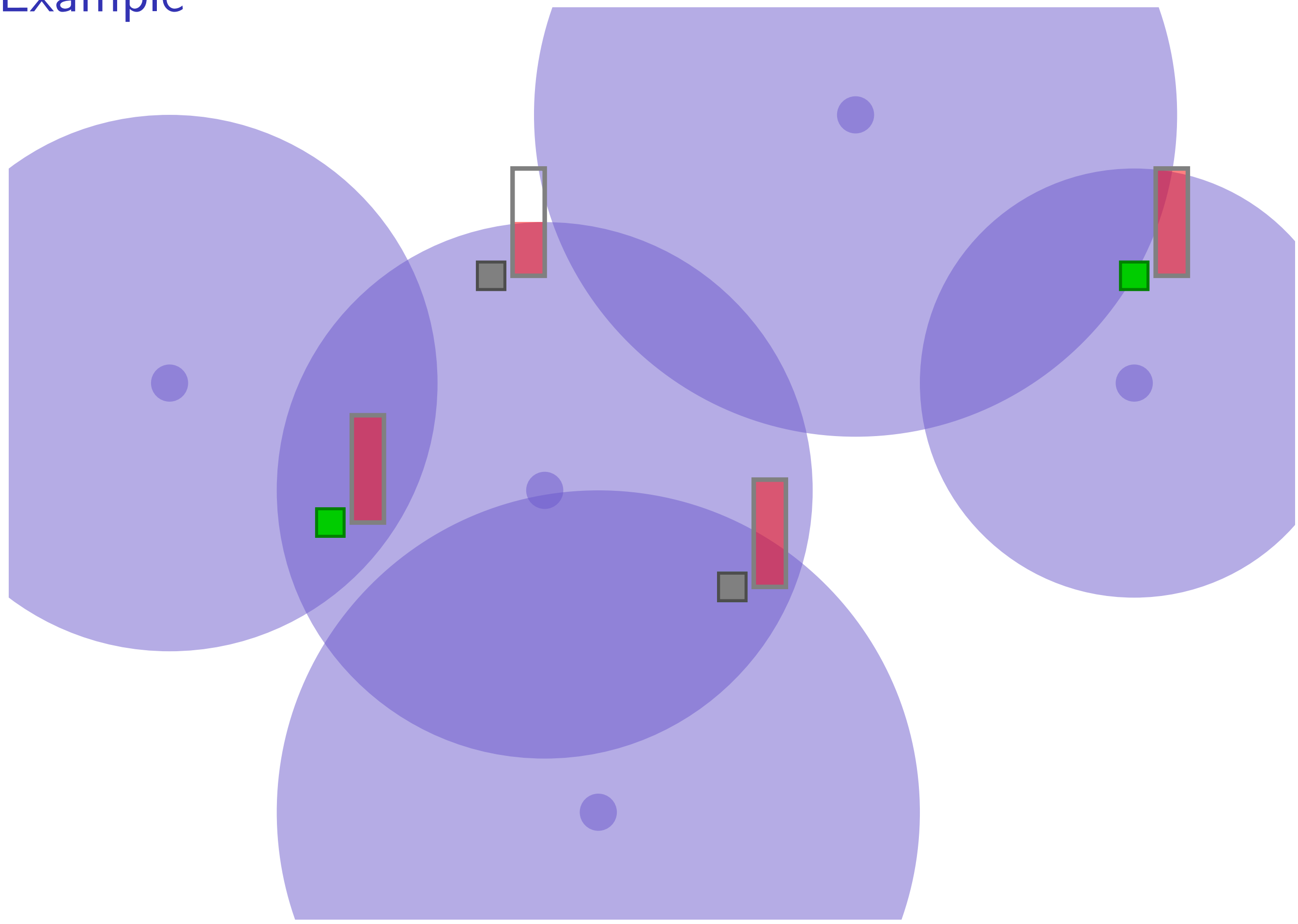
Example



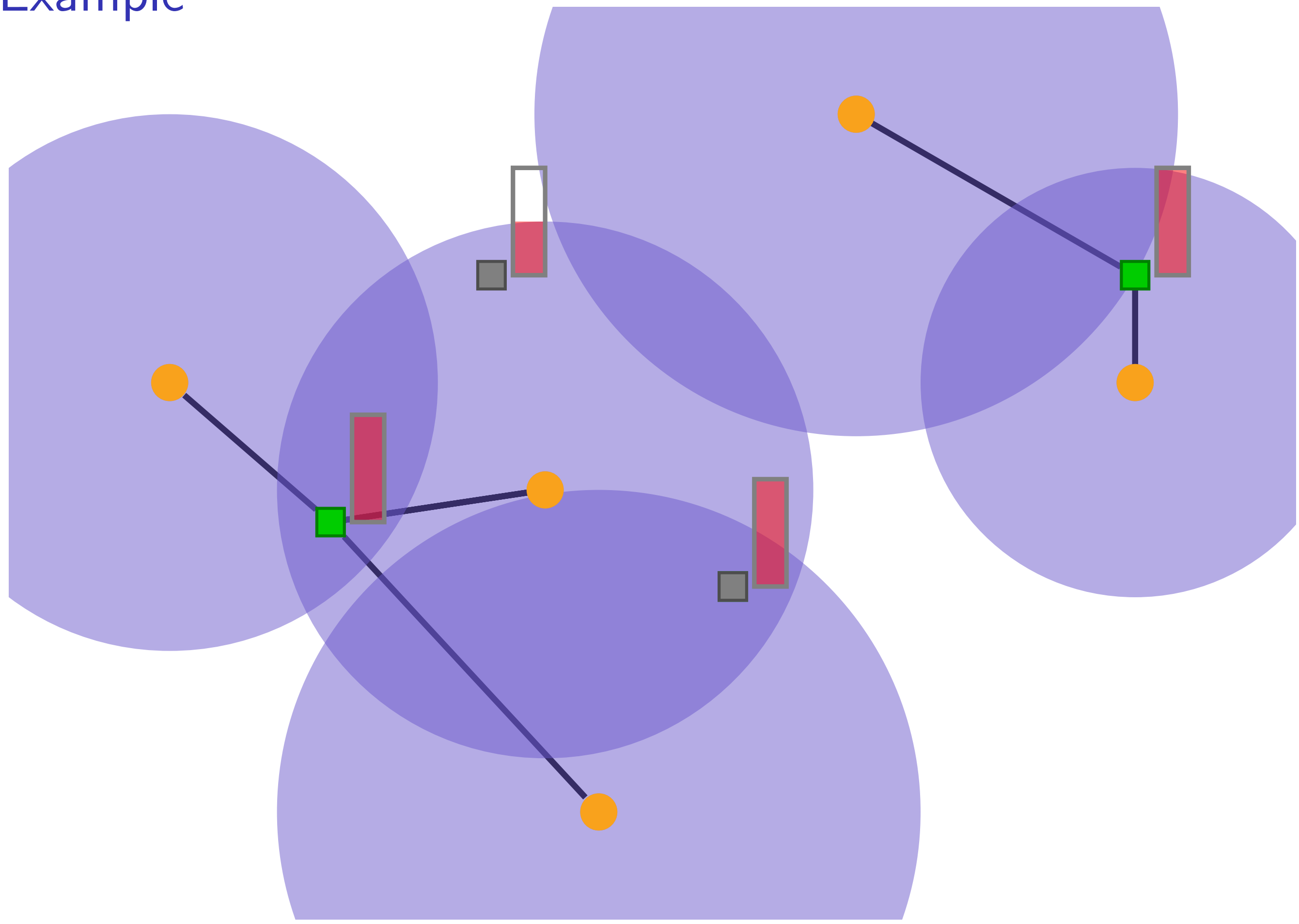
Example



Example



Example



Analysis

Lemma 7.10.

If a client j does not have a neighbor in F', then there is a facility $i \in F'$ such that $c_{ij} \leq 3v_j$.

Proof:... □

Theorem 7.11.

The algorithm is a 3-approximation algorithm for the uncapacitated facility location problem.

Proof:... □