Steiner Forest Problem

Given: Graph $G = (V, E)$ with costs $c_e \geq 0$, $e \in E$; k pairs $s_i, t_i \in V$.

Task: Find $F \subseteq E$ minimizing $c(F)$ and connecting s_i and t_i, for all i.

IP formulation: (let $\mathcal{S}_i := \{S \subseteq V \mid |S \cap \{s_i, t_i\}| = 1\}$ and $\mathcal{S} := \bigcup_{i=1}^{k} \mathcal{S}_i$)

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 1 && \text{for all } S \in \mathcal{S}, \\ & x_e \in \{0, 1\} && \text{for all } e \in E. \end{aligned}$$

Dual of LP relaxation ($x \geq 0$):

$$\begin{aligned} \max \quad & \sum_{S \in \mathcal{S}} y_S \\ \text{s.t.} \quad & \sum_{S \in \mathcal{S}:\, e \in \delta(S)} y_S \leq c_e && \text{for all } e \in E, \\ & y_S \geq 0 && \text{for all } S \in \mathcal{S}. \end{aligned}$$

Primal-Dual Algorithm for Steiner Forest Problem

1. set $y :\equiv 0$ and $F := \emptyset$;
2. while not all s_i-t_i pairs are connected in (V, F)
3. let C connected comp. of (V, F) with $|C \cap \{s_i, t_i\}| = 1$ for some i;
4. increase y_C until there is an $e \in \delta(C)$ with $\sum_{S \in \mathcal{S}:\, e \in \delta(S)} y_S = c_e$;
5. set $F := F \cup \{e\}$;

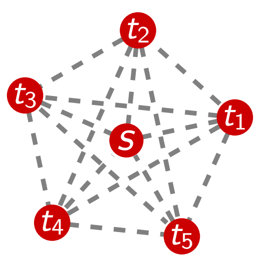
Analysis:

$$\sum_{e \in F} c_e = \sum_{e \in F} \sum_{S \in \mathcal{S}:\, e \in \delta(S)} y_S = \sum_{S \in \mathcal{S}} |\delta(S) \cap F| \cdot y_S$$

Problem:

It can happen that $|\delta(S) \cap F| = k$ for $y_S > 0$ and $\sum_{S \in \mathcal{S}} y_S \leq \frac{1}{k}\text{OPT}$:

- $G = K_{k+1}$ (complete graph)
- $s_i := s$ for $i = 1, \ldots, k$
- $c_e := 1$ for all $e \in E$



Primal-Dual Algorithm for Steiner Forest Problem

set $y :\equiv 0$ and $F := \emptyset$;
while not all s_i-t_i pairs are connected in (V, F)
    let C connected comp. of (V, F) with $|C \cap \{s_i, t_i\}| = 1$ for some i;
    increase y_C until there is an $e \in \delta(C)$ with $\sum_{S \in \mathcal{S}\colon e \in \delta(S)} y_S = c_e$;
    set $F := F \cup \{e\}$;

Analysis:

$$\sum_{e \in F} c_e = \sum_{e \in F} \sum_{S \in \mathcal{S}\colon e \in \delta(S)} y_S = \sum_{S \in \mathcal{S}} |\delta(S) \cap F| \cdot y_S$$

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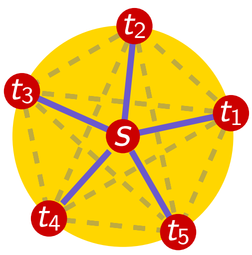


- $y_{\{s\}} = 1$
- $|\delta(\{s\}) \cap F| = k$
- $\sum_S y_S = 1$, $\mathrm{OPT} = k$

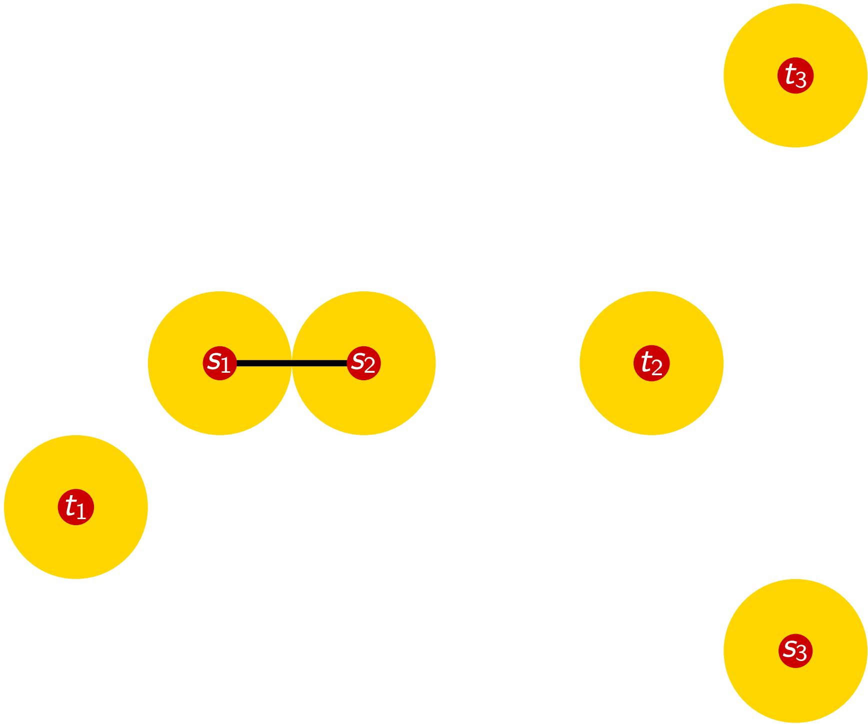
Refined Primal-Dual Algorithm for Steiner Forest Problem

set $y :\equiv 0$ and $F := \emptyset$;
while not all s_i-t_i pairs are connected in (V, F)
    let $\mathcal{C} := \{$conn. comp. C of (V, F): $|C \cap \{s_i, t_i\}| = 1$ for some $i\}$;
    increase y_C for all $C \in \mathcal{C}$ uniformly until for some $e \in \delta(C)$, $C \in \mathcal{C}$

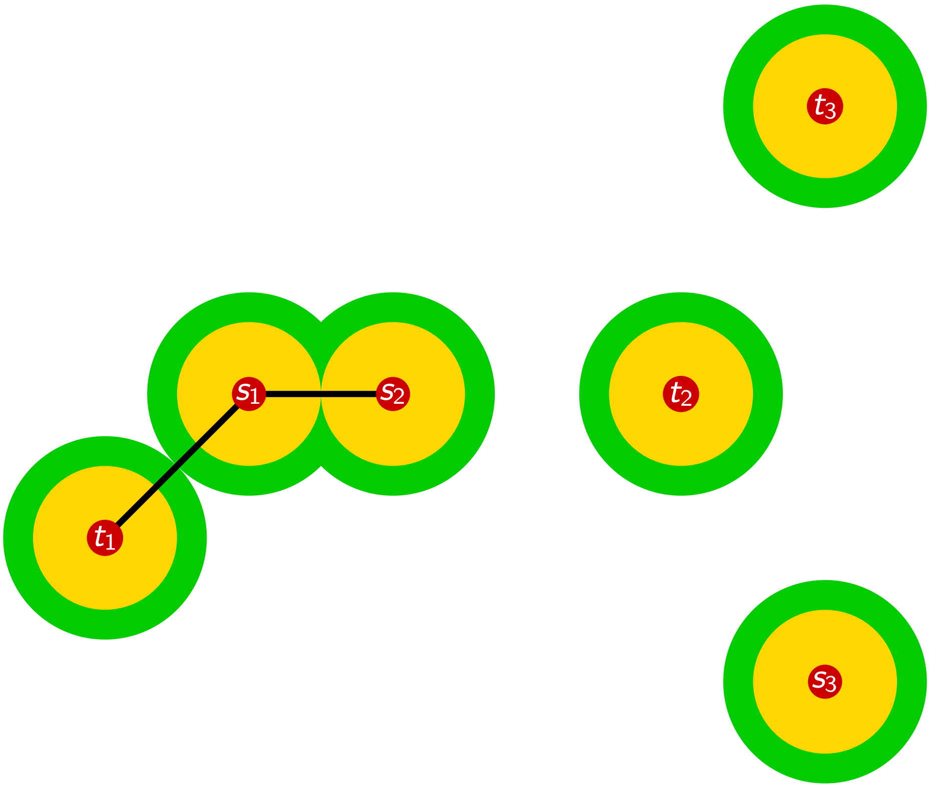
$$\sum_{S \in \mathcal{S}\colon e \in \delta(S)} y_S = c_e \;;$$

    set $F := F \cup \{e\}$;
consider edges $e \in F$ in reverse of the order in which they were added
    if $F \setminus \{e\}$ is a feasible solution, then remove e from F;

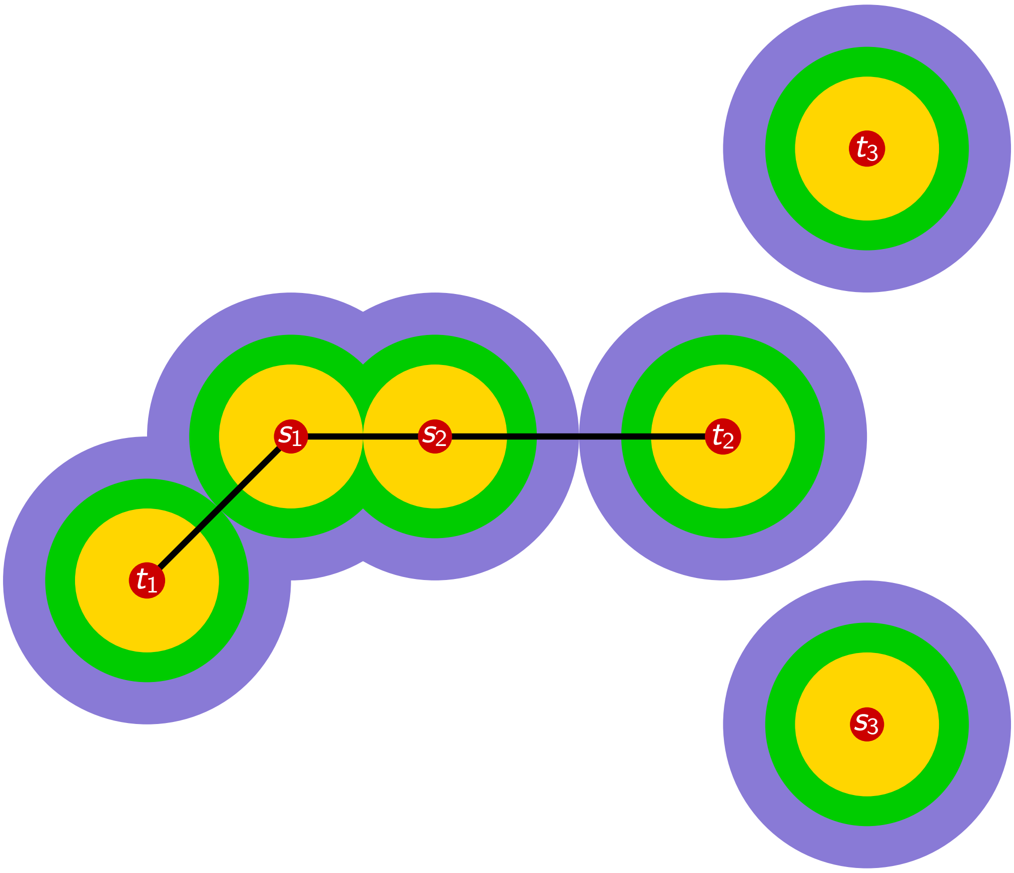
Example



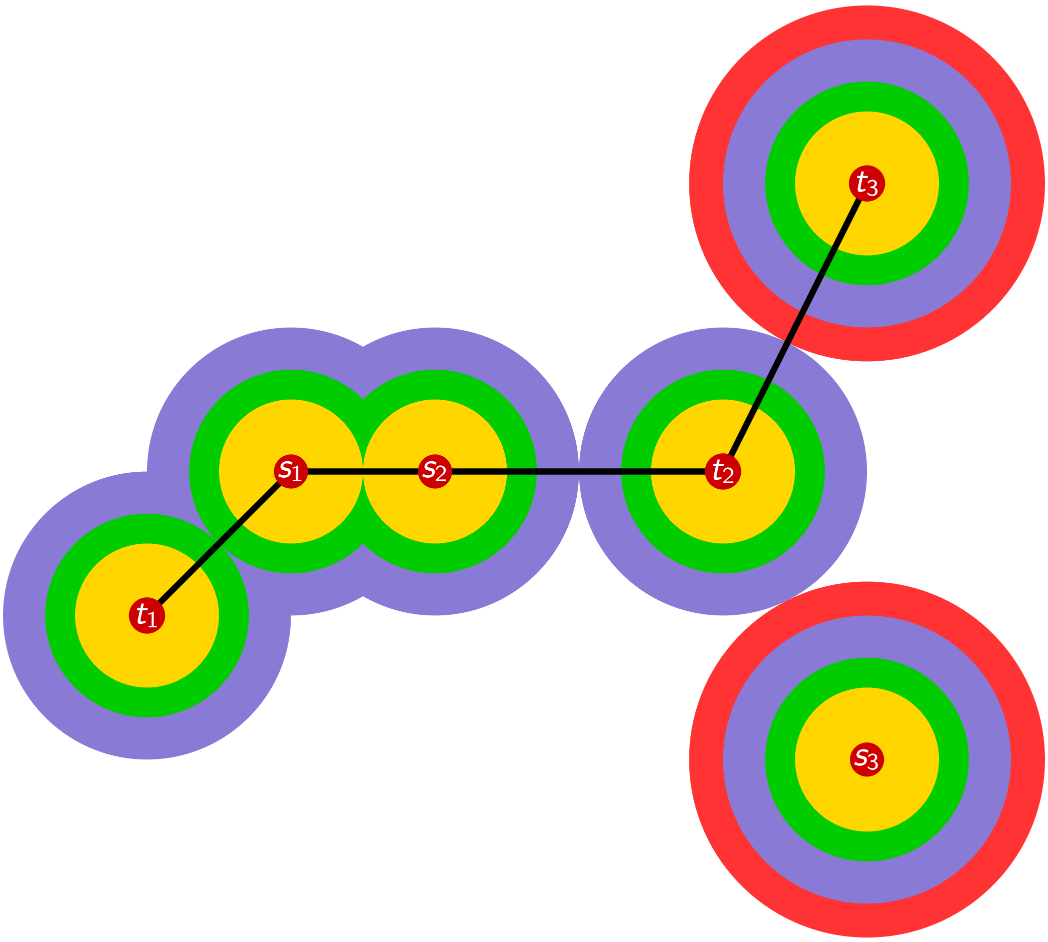
Example



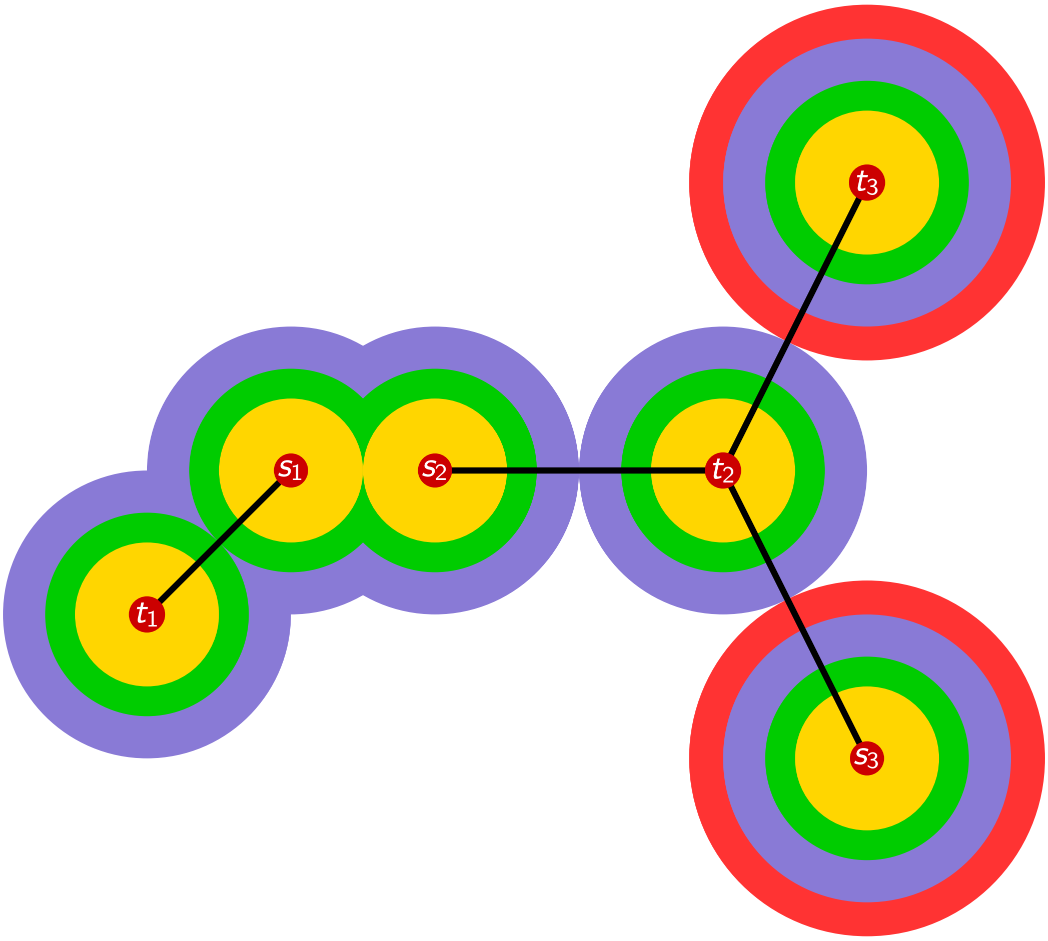
Example



Example



Example



Analysis

Observation. At any point in the algorithm, the set of edges F is a forest.

Lemma 7.5.

Let F' be the final set of edges returned by the algorithm. For any $\mathcal{C}$ in any iteration of the algorithm,

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}| \ .$$

Proof:. . . □

Theorem 7.6.

The refined primal-dual algorithm is a 2-approximation algorithm for the generalized Steiner Tree Problem.

Proof:. . . □

Corollary 7.7.

Integrality gap of the LP relaxation is at most 2. This bound is tight. □