Coloring Dense 3-Colorable Graphs

Definition 5.24.

i. A graph on n nodes is dense if for some constant $\alpha > 0$ the number of edges is at least $\alpha \cdot \binom{n}{2}$.
ii. For $0 \leq \delta < 1$, a graph on n nodes is δ-dense if every node has at least $\delta \cdot n$ neighbors.

Remark: Any δ-dense graph is dense.

We state the following Theorem without proof.

Theorem 5.25.

a. It is NP-hard to decide if a graph can be colored with only 3 colors, or needs at least 5 colors.
b. Assuming a variant of the *Unique Games Conjecture*, for any constant $k > 3$, it is NP-hard to decide if a graph can be colored with only 3 colors, or needs at least k colors. □

Random Sampling

Theorem 5.26.

With high probability, we can properly color any δ-dense 3-colorable graph.

Random Sampling Algorithm

1. select random subset $S \subseteq V$ of $O(\ln n/\delta)$ nodes such that each node in $V \setminus S$ has a neighbor in S;
2. enumerate all possible 3-colorings of S;
3. try to extend each 3-coloring of S to a 3-coloring of G;

Lemma 5.27.

With high probability, the subset S in Step 1 can be obtained by including each node with probability $(3c \ln n)/(\delta n)$ for some constant c.

Proof:. . . □

Lemma 5.28.

Let $X_1, \ldots, X_n$ independent 0-1 random variables. For $X := \sum_{i=1}^{n} X_i$ and $\mu \leq \mathrm{E}[X]$ it holds that $\Pr[X = 0] < e^{-\mu}$. □

Chapter 6: Randomized Rounding of Semidefinite Programs

(cp. Williamson & Shmoys, Chapter 6)

Semidefinite Matrices

Definition 6.1.

A matrix $X \in \mathbb{R}^{n \times n}$ is positive semidefinite if $y^T \cdot X \cdot y \geq 0$ for all $y \in \mathbb{R}^n$. In this case we write $X \succeq 0$.

Theorem 6.2.

For symmetric matrix $X \in \mathbb{R}^{n \times n}$ the following statements are equivalent:

i. X is positive semidefinite;
ii. all eigenvalues of X are non-negative;
iii. $X = V^T \cdot V$ for some $V \in \mathbb{R}^{m \times n}$ where $m \leq n$;
iv. $X = \sum_{i=1}^{n} \lambda_i \, (w_i \cdot w_i^T)$ for some $\lambda_i \geq 0$ and $w_i \in \mathbb{R}^n$ such that $w_i^T \cdot w_i = 1$ and $w_i^T \cdot w_j = 0$ for $i \neq j$. □

Semidefinite Programs (SDPs)

Definition 6.3.
A semidefinite program is a linear program with the additional constraint that a square symmetric matrix of variables must be positive semidefinite.

Example.

$$
\begin{aligned}
\min / \max \quad & \sum_{i,j} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{ij} a_{ijk} x_{ij} = b_k && \text{for all } k, \\
& x_{ij} = x_{ji} && \text{for all } i, j, \\
& X = (x_{ij}) \succeq 0
\end{aligned}
$$

Remark. The set of feasible solutions of a semidefinite program is convex.

Vector Programs

A semidefinite program can be stated equivalently as a vector program and vice versa (see Theorem 6.2 iii):

$$
\begin{aligned}
\min / \max \quad & \sum_{i,j} c_{ij} \left(v_i^T \cdot v_j\right) \\
\text{s.t.} \quad & \sum_{ij} a_{ijk} \left(v_i^T \cdot v_j\right) = b_k && \text{for all } k, \\
& v_i \in \mathbb{R}^n && \text{for all } i = 1, \ldots, n.
\end{aligned}
$$

Remark.

- Under mild technical conditions, semidefinite programs can be solved within additive error ε in time polynomial in input size and $\log(1/\varepsilon)$.
- For simplicity, we assume in the following that we can efficiently obtain an optimal solution.