Minimizing the Weighted Sum of Completion Times

Given: jobs with processing time $p_j \in \mathbb{Z}_{>0}$, weight $w_j \geq 0$, and release date $r_j \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, n$.

Task: Schedule the jobs nonpreemptively on a single machine; minimize the total weighted completion time $\sum_{j=1}^{n} w_j \cdot C_j$.

Let $T := \max_j r_j + \sum_{j=1}^{n} p_j$ (upper bound on all completion times).

Consider an integer programming relaxation with variables

$$y_{jt} = \begin{cases} 1 & \text{if job } j \text{ is processed in time } [t-1, t), \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, n$, $t = 1, \ldots, T$.

Integer Programming Relaxation

$$\begin{aligned}
\min \quad & \sum_{j=1}^{n} w_j \cdot C'_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} y_{jt} \leq 1 && \text{for } t = 1, \ldots, T, \\
& \sum_{t=1}^{T} y_{jt} = p_j && \text{for } j = 1, \ldots, n, \\
& y_{jt} = 0 && \text{for } j = 1, \ldots, n,\ t = 1, \ldots, r_j, \\
& C'_j = \frac{1}{p_j} \sum_{t=1}^{T} y_{jt}\left(t - \tfrac{1}{2}\right) + \tfrac{1}{2} p_j && \text{for } j = 1, \ldots, n, \\
& y_{jt} \in \{0, 1\} && \text{for } j = 1, \ldots, n,\ t = 1, \ldots, T.
\end{aligned}$$

Remarks.

- Notice that in a feasible IP solution jobs might be preempted.
- In this case, C'_j underestimates the actual completion time of job j.

Randomized Rounding

1. compute optimal IP solution (y^*, C^*);
2. for $j = 1$ to n set random variable X_j to $t - \frac{1}{2}$ with probability y^*_{jt}/p_j;
3. sort the jobs such that $X_1 \leq X_2 \leq \cdots \leq X_n$;
4. schedule all jobs nonpreemptively and as early as possible in this order;

Lemma 5.16.
If the random variables X_j are independent, then $\mathrm{E}[C_j \mid X_j = x] \leq p_j + 2x$.

Proof:. . . □

Theorem 5.17.
The expected performance ratio of the randomized algorithm is at most 2.

Proof:. . . □

Computing an Optimum IP Solution

1. sort the jobs such that $w_1/p_1 \geq w_2/p_2 \geq \cdots \geq w_n/p_n$;
2. construct a preemptive schedule:
3.   always schedule the first available job which is not yet completed;
4. implicitely assign the variables y_{jt} (and C_j) accordingly;

Lemma 5.18.
The algorithm finds an optimal IP solution in polynomial time.

Proof: Exchange argument. . . □

Remarks.

- There are at most n combinatorially different choices of X_j.
- Randomized rounding can be implemented to run in polynomial time.
- Derandomization by method of conditional expectations.

Minimum-Capacity Multicommodity Flow Problem

Given: Undirected graph $G = (V, E)$ and k pairs $s_i, t_i \in V$, $i = 1, \dots, k$.

Task: Find single s_i-t_i-path in G, for $i = 1, \dots, k$.

Objective: Minimize maximum number of paths containing same edge.

Path-based IP formulation:

$$
\begin{array}{lll}
\min & W & \\
\text{s.t.} & \displaystyle\sum_{P \in \mathcal{P}_i} x_P = 1 & \text{for all } i = 1, \dots, k, \\
& \displaystyle\sum_{P : e \in P} x_P \le W & \text{for all } e \in E, \\
& x_P \in \{0, 1\} & \text{for all } P \in \mathcal{P}_i,\ i = 1, \dots, k.
\end{array}
$$

LP relaxation: Replace $x_P \in \{0, 1\}$ with $x_P \ge 0$.

- Despite exponential number of variables, LP relaxation can be solved in polynomial time!

Randomized Rounding

1. compute optimal LP solution (x^*, W^*);
2. for $i = 1$ to n
3. independently choose one path $P \in \mathcal{P}_i$ with probability x_P^*;

Theorem 5.19.
If $W^* \ge c \cdot \ln n$ for a large enough constant c, then with high probability, the total number of paths using any edge is at most $W^* + \sqrt{c \cdot W^* \ln n}$.

Proof:. . . □

Markov's Inequality and Chernoff Bound

Definition 5.20.

A probabilistic event happens with high probability if the probability that it does not occur is at most n^{-c} for some constant $c \geq 1$.

Lemma 5.21 (Markov's Inequality).

If $X \geq 0$ is a random variable, then $\Pr[X \geq a] \leq \mathrm{E}[X]/a$ for $a > 0$. □

Theorem 5.22 (Chernoff Bound).

Let $X_1, \dots, X_k$ be independent 0-1 random variables. Then for $X := \sum_{i=1}^{k} X_i$, $\mu \geq \mathrm{E}[X]$, and $0 < \delta \leq 1$

$$\Pr[X \geq (1+\delta)\cdot\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu} \leq e^{-\mu\cdot\delta^2/3} \ .$$

□

Performance Guarantees

Corollary 5.23.

a. If $W^* \geq c \cdot \ln n$, then randomized rounding with high probability produces a solution of value at most $2W^*$.
b. If $W^* \geq 1$, then with high probability the total number of paths using any edge is $O(\log n) \cdot W^*$.

Proof:. . . □