Non-linear Randomized Rounding

Consider a function $f : [0,1] \to [0,1]$.

1. compute an optimal solution (y^*, z^*) to the LP relaxation;
2. for $i = 1$ to n do
3. set x_i to true independently at random with probability $f(y_i^*)$;

Theorem 5.11.

Let $f : [0,1] \to [0,1]$ with $1 - 4^{-x} \leq f(x) \leq 4^{x-1}$ for all $x \in [0,1]$. Then non-linear randomized rounding with function f is a randomized 3/4-approximation algorithm.

Proof:. . . □

Remark:

- The integrality gap of the LP relaxation for MAX SAT is 3/4.
- Thus, 3/4 is best performance ratio one can prove based on the LP.

Randomized Algorithm for Prize-Collecting Steiner Trees

Idea:

- Obtain randomized variant of deterministic LP rounding algorithm from Chapter 4 by choosing α randomly.
- For some fixed $\gamma > 0$ choose α uniformly at random from $[\gamma, 1]$.
- That is, choose α from $[\gamma, 1]$ with constant density function $1/(1-\gamma)$.

Lemma 5.12.

The tree T returned by the randomized algorithm has expected cost

$$\mathrm{E}\left[\sum_{e \in E(T)} c_e\right] \leq \frac{2}{1-\gamma} \ln \frac{1}{\gamma} \sum_{e \in E} c_e \cdot x_e^* \ .$$

Proof:. . . □

Randomized Algorithm for Prize-Collecting Steiner Trees

Lemma 5.13.

The expected penalty costs are

$$\mathrm{E}\left[\sum_{i \in V \setminus V(T)} \pi_i\right] \leq \frac{1}{1-\gamma} \sum_{i \in V} \pi_i \cdot (1 - y_i^*) \ .$$

Proof:. . . □

Theorem 5.14.

For $\gamma := e^{-1/2}$ the expected cost of the solution is

$$\mathrm{E}\left[\sum_{e \in E(T)} c_e + \sum_{i \in V \setminus V(T)} \pi_i\right] \leq \frac{1}{1 - 1/\sqrt{e}} \cdot \mathrm{OPT}_{LP} \ .$$

Thus, we have a randomized 2.54-approximation algorithm. □

Derandomization and Integrality Gap

Derandomization.

- There are at most $n := |V|$ distinct values of y_i^*.
- Consider n sets $U_j := \{i \in V \mid y_i^* \geq y_j^*\}$.
- Any possible value of α corresponds to one of these n sets.
- Derandomize by trying each set U_j and choosing the best solution.

Integrality gap.

- There exist instances with integrality gap $2 - \frac{2}{n}$.
- By Theorem 5.14 that the integrality gap is at most $\frac{1}{1-1/\sqrt{e}} \approx 2.54$.
- We will prove later that the integrality gap is at most 2.

Randomized Algorithm for Uncapacitated Facility Location

In Chapter 4 we obtained an LP-based 4-approximation algorithm which computes a solution of cost at most

$$\sum_{i \in F} f_i \cdot y_i^* + 3 \cdot \sum_{j \in D} v_j^* \enspace .$$

Notation.

Let $C_j^* := \sum_{i \in F} c_{ij} \cdot x_{ij}^*$ denote the assignment cost of j paid by the LP, i. e.,

$$\mathrm{OPT}_{LP} = \sum_{i \in F} f_i \cdot y_i^* + \sum_{j \in D} C_j^* \enspace .$$

Idea:

- Include the assignment cost C_j^* in the analysis.
- Instead of bounding only the facility cost by OPT_{LP}, bound both the facility cost and part of the assignment cost by OPT_{LP}.

Randomized Algorithm for Uncapacitated Facility Location

Randomized algorithm for Uncapacitated Facility Location Problem

1. compute optimal LP solutions (x^*, y^*) and (v^*, w^*);
2. while $D \neq \emptyset$
3. choose $j := \mathrm{argmin}_{j' \in D}(v_{j'}^* + C_{j'}^*)$;
4. choose $i \in N(j)$ according to probability distribution x_{ij}^*;
5. assign all unassigned clients in $N^2(j)$ to facility i;
6. set $D := D \setminus N^2(j)$;

Theorem 5.15.

The algorithm above is a randomized 3-approximation algorithm for the Uncapacitated Facility Location Problem.

Proof:. . . □