Derandomization: Method of Conditional Expectations

Basic Idea:

- Consider random decisions sequentially one after another.
- Take next decision deterministically optimizing the expected solution value assuming that all remaining decisions are taken randomly.

Example: Derandomized version of randomized MAX SATalgorithm

Let W denote the total weight of satisfied clauses in final solution.

```
for i = 1 to n
    if   E[W | x_1 = b_1, ..., x_{i-1} = b_{i-1}, x_i = true]
           ≥ E[W | x_1 = b_1, ..., x_{i-1} = b_{i-1}, x_i = false]
    then set b_i := true;
    else set b_i := false;
return x:=b;
```


Method of Conditional Expectations: Analysis

Theorem 5.5.
The value of the solution computed by the deterministic MAX SAT algorithm is at least the expected value of the randomized solution.

Proof:. . . □

Remarks.

- The crucial step of the derandomized algorithm is to compute the conditional expectations.
- Notice that $\mathrm{E}[W \mid x_1 = b_1, \ldots, x_i = b_i]$

$$= \sum_{j=1}^{m} w_j \cdot \Pr[C_j = \text{true} \mid x_1 = b_1, \ldots, x_i = b_i]$$

and $\Pr[C_j = \text{true} \mid x_1 = b_1, \ldots, x_i = b_i]$

$$= \begin{cases} 1 & \text{if } x_1 = b_1, \ldots, x_i = b_i \text{ satisfies } C_j, \\ 1 - 1/2^k & \text{else,} \end{cases}$$

where k is the number of remaining literals in clause C_j.

Flipping Biased Coins

We first restrict to MAX SAT instances with no negated unit clause.

Lemma 5.6.

If each x_i is independently set to true with probability $p > 1/2$, then the probability that a clause is satisfied is at least $\min\{p, 1 - p^2\}$.

Proof:. . . $\square$

Theorem 5.7.

For $1/2 < p \leq 1$ this gives a randomized $\min\{p, 1 - p^2\}$-approximation algorithm for MAX SAT. $\square$

Notice: For $p = (\sqrt{5} - 1)/2$ we get $\min\{p, 1 - p^2\} = (\sqrt{5} - 1)/2 \approx 0.618$.

Remark:

The initial assumption on the non-existence of negated unit clauses is not needed to get the randomized $(\sqrt{5} - 1)/2$-approximation algorithm!

Integer Programming Formulation for MAX SAT

For $j = 1, \ldots, m$ let $P_j := \{i \mid \text{literal } x_i \text{ occurs in } C_j\}$

and $N_j := \{i \mid \text{literal } \overline{x_i} \text{ occurs in } C_j\}$.

That is,
$$C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \overline{x_i}.$$

IP formulation:

$$\begin{aligned}
\max \quad & \sum_{j=1}^{m} w_j \cdot z_j \\
\text{s.t.} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j && \text{for all } j = 1, \ldots, m, \\
& y_i \in \{0, 1\} && \text{for all } i = 1, \ldots, n, \\
& 0 \leq z_j \leq 1 && \text{for all } j = 1, \ldots, m.
\end{aligned}$$

LP relaxation: Replace $y_i \in \{0, 1\}$ with $0 \leq y_i \leq 1$ for all $i = 1, \ldots, n$.

Randomized Rounding

1. compute an optimal solution (y^*, z^*) to the LP relaxation;
2. for $i = 1$ to n do
3. set x_i to true independently at random with probability y_i^*;

Theorem 5.8.

Randomized rounding gives a randomized $(1 - 1/e)$-approximation algorithm for MAX SAT.

Proof:... □

Remark.

Algorithm can be derandomized by method of conditional expectations.

Choosing the Better of Two Solutions

Theorem 5.9.

Running either the unbiased randomized 1/2-approximation algorithm or the randomized rounding algorithm, both with probability 1/2, yields a randomized 3/4-approximation algorithm.

Proof:... □

Derandomizing the initial coin flip yields:

Corollary 5.10.

Running both algorithms and choosing the better of the two solutions is a randomized 3/4-approximation algorithm. □