Chapter 5: Random Sampling and Randomized Rounding of Linear Programs

(cp. Williamson & Shmoys, Chapter 5)

Randomized Approximation Algorithm

Definition 5.1.

A randomized α-approximation algorithm is a polynomial-time randomized algorithm which always finds a feasible solution whose *expected* value is bounded by $\alpha \cdot \mathrm{OPT}$.

Remarks

- Often, a randomized α-approximation algorithm can be *derandomized*, i. e., turned into a deterministic α-approximation algorithm.
- It is usually simpler to state and analyze the randomized algorithm.
- Sometimes, the only known way of analyzing a deterministic approximation algorithm is to analyze a randomized version.
- Sometimes one can show that the performance guarantee of a randomized algorithm holds with high probability.

Maximum Satisfiability Problem (MAX SAT)

Given: Boolean variables $x_1, \dots, x_n$ and clauses $C_1, \dots, C_m$ with weights $w_1, \dots, w_m \in \mathbb{R}_{\geq 0}$.

(Clause is disjunction of Boolean variables or negations, e. g., $x_1 \vee \overline{x_2} \vee x_3$)

Task: Find a truth assignment to $x_1, \dots, x_n$.

Objective: Maximize the total weight of satisfied clauses.

Example: $(x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_2 \vee x_3) \wedge (\overline{x_3})$

Remarks:

- A variable x_i or its negation $\overline{x_i}$ is a literal.
- The number of literals ℓ_j in clause C_j is its size or length.
- If $\ell_j = 1$, then C_j is a unit clause.
- W.l.o.g. no literal is repeated in a clause and clauses are distinct.
- W.l.o.g. at most one of x_i and $\overline{x_i}$ appears in a clause.

Randomized Truth Assignment

Theorem 5.2.

a. Setting each x_i to true independently with probability $1/2$ gives a randomized $1/2$-approximation algorithm for MAX SAT.

b. If $\ell_j \geq k$ for all $j = 1, \dots, m$, then the above algorithm is a randomized $(1 - 1/2^k)$-approximation algorithm.

Proof:. . . □

Maximum Exactly 3SAT (MAX E 3SAT): The special case of MAX SAT where $\ell_j = 3$ for all $j = 1, \dots, m$ is called MAX E 3SAT.

We state the following theorem without proof.

Theorem 5.3.

Unless $P = NP$, there is no $(7/8 + \varepsilon)$-approximation algorithm for MAX E 3SAT for any constant $\varepsilon > 0$.

Maximum Cut Problem (MAX CUT)

Given: Undirected Graph $G = (V, E)$ with edge weights $w_e \geq 0$, $e \in E$.

Task: Find $S \subset V$ maximizing $\sum_{e \in \delta(S)} w_e$.

Theorem 5.4.

Placing each node $v \in V$ into S independently at random with probability $1/2$ gives a randomized $1/2$-approximation algorithm for MAX CUT.

Proof:. . . □