Uncapacitated Facility Location Problem

Given: Set of facilities F with opening costs $f_i \geq 0$, $i \in F$;
set of clients D with connection costs $c_{ij} \geq 0$, $i \in F$, $j \in D$.

Task: Choose $F' \subseteq F$ and assign each client to nearest facility in F'.

Objective: Minimize $\sum_{i \in F'} f_i + \sum_{j \in D} \min_{i \in F'} c_{ij}$.

Remarks:

- This is a generalization of the Set Cover Problem.
- In the following, we consider the special case with metric costs c_{ij}.

IP formulation:

$$
\begin{aligned}
\min \quad & \sum_{i \in F} f_i \cdot y_i + \sum_{i \in F, j \in D} c_{ij} \cdot x_{ij} \\
\text{s.t.} \quad & \sum_{i \in F} x_{ij} = 1 && \text{for all } j \in D, \\
& y_i - x_{ij} \geq 0 && \text{for all } i \in F, j \in D, \\
& x_{ij}, y_i \in \{0, 1\} && \text{for all } i \in F, j \in D.
\end{aligned}
$$

LP Relaxation and Dual LP

$$
\begin{aligned}
\min \quad & \sum_{i \in F} f_i \cdot y_i + \sum_{i \in F, j \in D} c_{ij} \cdot x_{ij} \\
\text{s.t.} \quad & \sum_{i \in F} x_{ij} = 1 && \text{for all } j \in D, \\
& y_i - x_{ij} \geq 0 && \text{for all } i \in F, j \in D, \\
& x_{ij}, y_i \geq 0 && \text{for all } i \in F, j \in D.
\end{aligned}
$$

dual LP:

$$
\begin{aligned}
\max \quad & \sum_{j \in D} v_j \\
\text{s.t.} \quad & \sum_{j \in D} w_{ij} \leq f_i && \text{for all } i \in F, \\
& v_j - w_{ij} \leq c_{ij} && \text{for all } i \in F, j \in D, \\
& w_{ij} \geq 0 && \text{for all } i \in F, j \in D.
\end{aligned}
$$

Interpretation of the dual LP:

- v_j is the total amount that customer j wants to pay for being served.
- customer j might contribute w_{ij} to facility i for being connected to i.

Structure of Optimal LP Solution

Let (x^*, y^*) and (v^*, w^*) be optimal solutions to the primal and dual LP, respectively.

Notation:

- Facility i *neighbors* client j if $x^*_{ij} > 0$; $N(j) := \{i \in F \mid x^*_{ij} > 0\}$.
- $N^2(j) := \{\ell \in D \mid \text{client } \ell \text{ neighbors some facility } i \in N(j)\}$.

Lemma 4.10.

If clients $j_1, \ldots, j_k$ have disjoint neighborhoods $N(j_1), \ldots, N(j_k)$, then opening cheapest facility in each neighborhood costs $\leq \sum_{i \in F} f_i \cdot y^*_i \leq \text{OPT}$.

Proof:. . . □

Lemma 4.11.

For each client j, $v^*_j \geq c_{ij}$ for all $i \in N(j)$.

Proof: Follows from complementary slackness. □

Deterministic LP Rounding Algorithm

1. compute optimal LP solutions (x^*, y^*) and (v^*, w^*);
2. while $D \neq \emptyset$
3. choose $j := \text{argmin}_{j' \in D} v^*_{j'}$ and $i := \text{argmin}_{i' \in N(j)} f_{i'}$;
4. assign all unassigned clients in $N^2(j)$ to facility i;
5. set $D := D \setminus N^2(j)$;

Theorem 4.12.

The algorithm above is a 4-approximation algorithm.

Proof:. . . □

We finally mention the following non-approximability result without proof.

Theorem 4.13.

There is no α-approximation algorithm for the metric uncapacitated facility location problem with $\alpha < 1.463$ unless each problem in NP has an $O(n^{O(\log \log n)})$ time algorithm.

Bin Packing Revisited

In the previous chapter we showed how to find a solution to instance I with at most $(1+\varepsilon)\text{OPT}(I)+1$ bins in polynomial time.

Goal: Use at most $\text{OPT}(I) + O(\log^2 \text{OPT}(I))$ bins!

Ingredients:

- Replace dynamic program with integer program plus LP rounding.
- Improved grouping scheme.
- Recursive application of two previous ingredients.

Notice:
By Lemma 3.14 we can assume that all items have size at least $1/\text{SIZE}(I)$.